SUITABLE MEASURES OF JUMPS IN STOCHASTIC MODELS FOR STOCK MARKET INDICES

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Dedication

To the Almighty God and to the African woman in scientific research faced with multiple tasks.

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Abstract

The dynamics of the stock market indices log-returns ($\Delta(ln\tilde{S}_t)$) have been characterised by non-normal features such as upward and downward jumps of different measures, asymmetric and leptokurtic features. The Bi-Power Variation (BPV) process has been used to develop jump-estimators to detect jumps in ($\Delta(ln\tilde{S}_t)$. However, the existing jump-estimators are restricted to the BPV case of the Realised Multi-Power Variation (RMPV) processes, and existing models do not accommodate these features. Therefore, this study was designed to construct unrestricted Particular Higher-Order Cases (PHOC) jump-estimators, and build suitable models that can accommodate the jumps and non-normal features found in ($\Delta(ln\tilde{S}_t)$.

The limits in probability and distribution were used to derive the Jump Test Models (JTM) in the PHOC of the RMPV processes. The JTM were used to test for jumps under the null hypothesis (H_0) of no jump in ($\Delta(ln\tilde{S}_t)$ at a 5% level of significance in three stock markets, namely: Nigerian, UK, and Japan. These were used to build the dynamics of ($\Delta(ln\tilde{S}_t)$. The convolution of densities and the Lévy-Itô decomposition methods were used to derive the Probability Density Functions (PDF) and the Lévy-Khintchine (LK) formulae of two novel skewed models. The maximum likelihood estimation method was used to estimate the optimal values of the parameters in the models. The Kolmogorov-Smirnov, Anderson-Darling statistics, and the basic moments were used to test the suitability of these models to the empirical stock market data and compared with three existing models viz: Black-Scholes (BS), normal and double-exponential jump-diffusion models.

The JTM derived for the PHOC of the RMPV processes were:
$\hat{\mathrm{Z}}_{\mathrm{m}} = \Delta^{-0.5}\left(\frac{\mu_{2/m}^{-m}\{X\}_{\Delta,t}^{[r_1,\dots,r_m]}}{\{X\}_{\Delta,t}^{[2]}} - 1\right)\left(\varphi_{RMPV}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}\right)^{-1}$, for $m = 2\cdots 10$, where, $\{X\}_{\Delta,t}^{[r_1,\dots,r_m]}$, $\{X\}_{\Delta,t}^{[2]}$, φ_{RMPV}, $\hat{p}$ and $\hat{q}$ are the RMPV, realised variance, asymp-

totic variance, estimators of bi-power and quad-power variation, respectively. Jumps in $\Delta(ln\tilde{S}_t)$ were observed and H_0 was rejected. The dynamics of $\Delta(ln\tilde{S}_t)$ was derived as: $\Delta(ln\tilde{S}_t) = (\mu - \frac{1}{2}\sigma^2)\Delta + \sigma\Delta W_t + J(Q_j^u)\Delta N_t^u + J(Q_j^d)\Delta N_t^d$, where,$\mu$, σ, W_t, $J(Q_j^u)$, $J(Q_j^d)$, N_t^u and N_t^d are respectively drift and volatility parameters, standard Brownian motion, upward and downward jump measures with intensities λ_j^u and λ_j^d, respectively. The PDF of the Asymmetric-Laplace (AL) and the Modified Double-Rayleigh (MDR) were models derived were: $f_{\Delta(\ln\tilde{S}_t)}(x) = \frac{(1-\lambda\Delta t)}{\sigma\sqrt{\Delta t}}\varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) +$

$$\Delta t\Bigg(p_\kappa\alpha_2\lambda_j^u exp\left(\frac{2\alpha_1\mu_j+\alpha_1\sigma^2\Delta t}{2}\right)exp\left(-\left(x-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\alpha_1\right)$$

$$\Phi_a\big(\mu_j\big) + q_k\alpha_2\lambda_j^d exp\left(\frac{2\alpha_2\mu_j+\alpha_2\sigma^2\Delta t}{2}\right)exp\left(-\left(x-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t\right)\alpha_2\right)\Phi_b\big(-\mu_j\big)\Bigg)$$

and

$$f_{\Delta(\ln\tilde{S}_t)x} = \frac{(1-\lambda\Delta t)}{\sigma\sqrt{\Delta t}}\varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) + \Bigg(p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)$$

$$\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\sqrt{\pi\vartheta}\Phi_a(\mu_j)-\mu_j\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right)-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)+\right.$$

$$\left.\theta\sqrt{\pi\hat{\vartheta}}\Phi_b(\mu_j) - \mu_j\sqrt{\pi\hat{\vartheta}}\Phi_b(-\mu_j)\right)\Bigg)\Delta t,$$

where,

$\theta = \frac{\sigma_j^u(\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}$, $\rho = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}$, $\vartheta = \frac{2\sigma^2\Delta t\sigma_j^u}{(\sigma_j^u+\sigma^2\Delta t)}$, $\hat{\theta} = \frac{\sigma_j^d(\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$, $\hat{\rho} = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$, $\hat{\vartheta} = \frac{2\sigma^2\Delta t\sigma_j^d}{(\sigma_j^d+\sigma^2\Delta t)}$, $\eta = \frac{\lambda_j^u}{(\sigma_j^u)}\varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$ and $\hat{\eta} = \frac{\lambda_j^d}{(\sigma_j^d)}\varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$. The derived LK formulae of the novel models were: $\psi(u) = iu\mu - \frac{1}{2}\sigma^2u^2 - \left(\frac{\lambda_j^u p_k\alpha_1}{\alpha_1-iu} + \frac{\lambda_j^d q_k\alpha_2}{\alpha_2+iu}\right)\mathrm{e}^{iu\mu_j} + \lambda_j^d q_k + \lambda_j^u p_k$ and $\psi(u) = iu\mu - \frac{1}{2}\sigma^2\mu^2 - \frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d} + \left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\mathrm{e}^{iu\mu_j}$, respectively.The optimal values of the parameters: $(\mu_d, \sigma, \alpha_1, \alpha_2, p_k, q_k, \lambda_j^u, \lambda_j^d, \mu_j)$ and $(\mu_d, \sigma, \sigma_j^u, \sigma_j^d, p, q\lambda_j^u, \lambda_j^d, \mu_j)$ in the models were obtained. The AL and MDR were models fit the empirical distributions better than the existing models, having the BS model in the worst-case scenario.

The jump test models of the particular higher-order cases were found to be better jump-estimators. The asymmetric-Laplace and modified double-Rayleigh jump-diffusion models proved more suitable for capturing jumps and non-normal features in the stock market indices log-returns.

Keywords: Stock indices, Jump-processes, Asymmetric-Laplace jump-diffusion , modified double-Rayleigh, Jump-estimators. ***Word count:*** 466

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CHAPTER ONE

INTRODUCTION

1.1 Background to the study

The suitability of a model, which represents the real pattern of its random process, is its ability to capture almost all the features associated with the system's uncertainties. For example, a deterministic model may not fit into any such chaotic real-life process since its predictions are with full assurance. A stochastic model, therefore, may give a better representation of such processes with unforetold uncertainties.

The stock price $\tilde{S}_t$ at time t, has been found to be one of the most unstable variables (Fama, 1965, 1995) in a stock market; its instability results from sudden changes that occur very frequently and randomly in the market, everyday. This has been a major concern for most investors who would want to know their fate when they invest in stocks. Over the years, researchers have looked into obtaining models that could best describe stock price behaviour, in order to advise investors and owners of corporations looking for convenient ways to raise money (Adeosun *et al.*, 2015) via stock investment.

The Scottish Botanist, Robert Brown (as stated in Bachelier, 1900) observed the random collision of some tiny particles with the molecules of the liquids; he introduced what is called the **Brownian motion (BM)**. The mathematics of the Brownian motion, which include the derivation of equations of the BM, existence of the BM and the Weiner measure that gives the probability distribution of the BM were given by Albert Einstein and Norbert Wiener, in Akyildirim and Soner (2014). Financial modelling under the category of stochastic stock price models began with Bachelier's work as also recorded in Akyildirim and Soner (2014). The Bachelier model assumes that the dynamics of a stock price process $\tilde{S}_t$, obeys the

Stochastic Differential Equation (SDE):

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t \tag{1.1}$$

where, σ is the volatility of the stock price and W_t is a standard Brownian motion. It was later observed by Osborne (1952), that ***Bachieler's model gave rise to negative stock price value almost surely,*** which contradicts the real life experience in the market. An improvement of Bachieler's model is Osborne's model, which was later modified by Samuelson (1965), who introduced the ***Geometric Brownian Motion (GBM) model***. The dynamics of the GBM model was described by Samuelson as:

$$d\tilde{S}_t = \tilde{S}_t\big(\mu dt + \sigma dW_t\big) \tag{1.2}$$

where, μ is the drift (mean) parameter and σ, $\tilde{S}_t$ and W_t are as defined above. A model, known as the ***Black-Scholes (B-S) model*** used for pricing Options which depends on the assumptions of the GBM model was introduced by Black and Scholes (1973a).

Despite the wide acceptance of the B-S model, it was later found not to be consistent with observed market prices (Schoutens, 2003). This inconsistency results from its unrealistic assumptions that the log returns of observed stocks prices are normally distributed (which they are not), rather, they are in most cases ***negatively skewed*** and ***leptokurtic*** (see also Trautmann and Beinent (1994), and Adeosun *et al.* (2016) all of which contradict the real-life observed price movements. The ***asymmetric property (skewness)*** is always different from zero, in most cases, it gives rise to ***negative values*** which imply ***longer tail to the left than the right***. The frequent and large movements in most stock price processes result in ***excess kurtosis,*** and their **paths exhibit jumps of different measures**. More so, the market crash that occurred in 1997 resulted in huge investment loss as a result of ***sudden downward jumps*** in market price. As suggested by Kou (2002), ***a suitable model should capture some important empirical features that are found in the real-life financial data***.

Attempts have been made by researchers to modify the B-S model in order to capture the above-listed features, in the sense that new models are obtained by the addition of a jump term to the GBM in equation (1.2) above. These are the jump-diffusion models according to Merton (1976), Duffie *et al.*(2000), Hanson and Westman (2002), Kou (2002), Synowiec (2008) and Lau *et al.*(2019); they have it that the dynamics of the stock price process in its generalised form is assumed to be:

$$d\tilde{S}_t = \tilde{S}_t\big(\mu dt + \sigma dW_t + J(Q_j)dN_t\big) \tag{1.3}$$

where, $J(Q_j)$ is a random jump process, and N_t is a discontinuous one-dimensional standard Poisson process with jump rate λ, such that:

$$\mathbb{P}(N_t = k) = \frac{exp(\lambda\Delta t)(\lambda\Delta t)^k}{k!} \tag{1.4}$$

The difference in the works of Merton (1976), Duffie *et al.*(2000), Hanson and Westman (2002), Kou (2002), Synowiec (2008) and Lau *et al.* (2019); is found to be as a result of different assumed measures of $J(Q_j)$, which could be symmetric or asymmetric in some cases. The Merton and Kou models for example, have addressed the leptokurtic property of $J(Q_j)$. The first assumes a symmetric normal distribution for $J(Q_j)$, and the second assumes asymmetric double exponential distribution for $J(Q_j)$. On the contrary, the distribution of the jumps that occur in any price process may not only be symmetric according to Lau *et al.* (2019), it could also be skewed in both the upward and downward jumps.

The economic activities in any nation's stock market are of utmost concern to marketers, investors, and the nation at large. In a stock market, ***the market index is a true picture of the overall performance of the market***, and it is an estimator that reflects the daily general market value. The **stock market indices** is a **major indicator of the value of the market**, since it is an ***instant measure*** used to ascertain the **direction of the market** as well as giving a picture of the ***overall performance of the market.*** The All Share Index (ASI) (for example) of the Nigeria Stock Market (NSM) is an example of a ***Market Index.*** The availability of stock market index data varies from one market to the other across the globe. High-frequency data are categorised into two types: the inter-day daily data of high volume and the intra-day data: minutes and hourly observations of high frequency (all of which are discrete observations). Processes modelling financial data can be categorised into three, depending on the empirical findings from the real market situation over a period of time. These include processes with continuous paths, discontinuous paths (jumps), and processes with both continuous and discontinuous paths. Therefore, researchers must verify the kind of stochastic process governing the underlying stock under study before making assertions. To buttress the above-mentioned, Bandi and Renǿ (2016) suggested that since most financial data exhibit jumps in their price processes, ***the suitability of a model is embedded in its ability to capture jumps in a price process***. To determine stock indices' dynamics, there is a need to examine and estimate some essential asymptotic features, the presence or absence of jumps in available discretely-observed price data.

In view of the above, the stock market indices was considered as a discrete observed process: $X = \{X_t\}_{t\geq 0}$ defined on any given interval [0, t] such that the observations are made for all discrete time $0 = t_0, t_1 \dots t_n = t$, where the j^{th} observed time is given as:

$$j\Delta = j\frac{t}{n}, \quad j = 1, 2, \cdots, n \tag{1.5}$$

where, n is the number of observations and Δ is the time interval between two successive observations, within $[0, t]$, which are assumed to be of equal distance. These observations are of utmost importance since they give the kind of stochastic process governed by an underlying stock. The challenge of capturing jumps in discretely-observed processes, in the face of available high-frequency data on the Internet (especially when n is so large that Δ is vanishingly small) is on the increase.

Given a positive real constant r, the r^{th} - order realised power variation of such a process is given as:

$$\{X_\Delta\}^{(r)} = \Delta^{1-r/2} \sum_{j=1}^{[t/\Delta]} \left| X_{j\Delta} - X_{(j-1)\Delta} \right|^r \tag{1.6}$$

where, $X_{j\Delta}$ is the j^{th} observed log return price for $j = 1, 2, \cdots, n$ and $[t/\Delta]$ is the finite integer before n. The r^{th} - order power variation process is given as the probability limit of the r^{th}-order realised power variation (RPV):

$$\{X\}^{(r)} = \mathbb{P} - \lim_{\Delta \to 0} \{X_\Delta\}^{(r)}. \tag{1.7}$$

An estimation of the Quadratic Variation (QV) defined by $[X]_t$ is obtained for $r = 2$ of the realised variation in equation (1.7). In Barndorff-Nielsen and Shephard (2003b), the Realised Variance (RV) was proven to be a consistent estimator (as $n \to \infty$) of $[X]_t$, when X is assumed to belong to a class of continuous stochastic volatility semimartingales (Svsm^c).

Given that $X_t \in Svsm^c$ (Barndorff- Nielsen and Shephard, 2003b, He *et al.*, 2018) such that:

$$X_t = \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s \tag{1.8}$$

where, $\int_0^t \alpha_s ds = A_t$ is an adapted, càdlàg process with **finite variation** which implies that the variation of each path $t \to A_t$ is bounded over each finite interval in $[0,t]$ and $\int_0^t \sigma_s dW_s = M_t$ is a local Martingale, which is continuous and an Itô integral of the spot volatility process $\sigma_t > 0$ with respect to a standard Brownian motion W_t; such that, the Integrated Volatility (IV) process is also assumed finite. Then, the limit distribution of functions of equation (1.7) above, its convergence in probability and the central limit theorem results were obtained in Barndorff-Nielsen *et al.* (2006b). Based on the procedure given in Barndorff-Neilsen *et al.* (2006b), a jump test method was achieved in Aït-Sahalia and Jacod (2009). The main limitation of the results of the RPV, subject to an $Svsm^c$ process is that when jumps are added to a class of models described in equation (1.8), the RV can no longer estimate the IV, instead, it gives a result of the sum of the IV and the QV of the jump component. Hence, the need for a robust process that cannot be affected when jumps are incorporated into the process.

The realised multipower variation process defined on a one-dimensional semi-martingale process in its generalised form is given as:

$$\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)} = \Delta^{1-\delta(r_1,\ldots,r_m)} \sum_{j=1}^{c(t,m,\Delta)} f(x_j, r_i). \tag{1.9}$$

as defined in Barndorff-Nielsen *et al.* (2006b), where $\delta(r_1,\ldots,r_m) = \frac{1}{2}\sum_{i=1}^m r_i$, $c(t,m,\Delta) = [{}^t/_\Delta] - (m-1)$ and $f(x_j, r_i) = \prod_{i=0}^{m-1} \left|x_{j+i}\right|^{r_{i+1}}$ for $n > m$. The asymptotic properties of equation (1.9) above, were extensively given in Barndorff-Neilsen *et al.* (2006b) and Kinnebrock and Podolskij (2008). Particular cases of equation (1.9) are the Bipower, Tripower and the Quadpower processes which can be found in Barndorff-Nielsen and Shephard (2006) and Ysusi (2006).

The BNS method for jump test named after **Barndorff-Nielsen and Shephard** was established in Barndorff-Nielsen and Shephard (2006) for $X_t \in Svsm^c$ subject to the above stated assumptions for the processes σ_t^2, α_t and W_t. This method was basically derived from the asymptotic distribution of the difference of the realised bipower variation process: $\{X\}_{\Delta,t}^{(1,1)}$ and the realised variance process $[X]_{\Delta,t}^{(2)}$. That is, for $m = 2$, *and* $r_1 = r_2 = 1$ in equation (1.9), then,

$$\frac{\Delta^{-0.5}\left(\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)} - [X]_{\Delta,t}^{(2)}\right)}{\sqrt{\int_0^t \sigma_s^4 ds}} \xrightarrow{L} N\left(0, \varphi_{BPV}\right) \tag{1.10}$$

where, φ_{BPV} is the asymptotic variance of the convergence in law (distribution) result given in equation (1.10) above, such that:

$$\varphi_{BPV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.6091, \mu_1 = \frac{\sqrt{2}}{\sqrt{\pi}} \tag{1.11}$$

One of the contributions in this thesis to the description in equations (1.10) and (1.11), as well as the work given in Barndorff-Nielsen and Shephard (2006) entails a derivation of the asymptotic theories for particular cases of the realised multipower variation ($RMPV$) process ***(the convergence in distribution (law) of the difference of the realised variance*** RV ***and particular cases of the*** $RMPV$ ***process.***). Hence, the ***asymptotic variances of the particular cases that is,*** φ_{RBV}, φ_{RTV}, φ_{QPV}, φ_{PPV}, φ_{HPV}, φ_{H_PPV}, φ_{OpV}, φ_{NPV} ***and*** φ_{DPV} respectively for the realised ***Bipower, Tripower, Quadpower, Pentpower, Hexpower, Heptpower, Octpower, Nonpower and Decpower variation processes*** were obtained. Based on the results obtained, jump test models from the asymptotic properties of the particular higher-order cases of the $RMPV$ process. These results are extensions of the results in the works of Barndorff-Nielsen and Shephard (2006), Barndorff-Nielsen *et al.* (2006a) and Ysusi (2006). These are the generalisation of the BNS jump test model for detecting jumps in discretely-observed data, and then ***suitable jump-diffusion models*** for the stock indices were suggested. Hence, a ***family of skewed jump-diffusion models with non-zero location parameters and scale parameters*** for upward and downward distributions of the random jump processes was considered. The novel models are: the Asymmetric Laplace jump-diffusion (ALJD) model (whose jump process obeys the Asymmetric Laplace (AL) distribution which is a skewed-family of the Laplace distribution, proposed by Kozubowski and Podgorski (2000) and Kotz *et al.* (2012) and the ***modified double Rayleigh jump-diffusion*** ($MDRJD$) ***model*** for the stock price indices.

1.2 Statement of the problem

Jumps are difficult to detect from discrete financial data (stock market indices are available in the discrete form), but useful in **asset pricing and risk management.** To suggest a suitable stochastic process that best describes stock indices data paths, it is necessary to ascertain the presence or absence of jumps in the discrete data. In this regard, many works have been done on jump test given discrete financial data, especially concerning the **realised quantities (realised power variation, realised**

bipower variation). The realised bipower variation is a consistent estimator of the integrated variance (Barndorff-Nielsen *et al.* (2006c), Barndorff-Nielsen and Shephard (2006), Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen *et al.*, 2006b) when jumps are present. However, **the existing jump-estimators are restricted to the bipower case of the realised multi-power variation processes, and owing to the increasingly volatile nature of the stock indices, more robust and better jump test models are required to capture these volatile features in the market.** A generalisation of the BNS jump test method and the particular cases of the higher order of the realised multipower variation for jump test have not been applied to stock indices in literature to the best of our knowledge.

Jump-diffusion processes applied to model stock indices have been noted in literature. Categorically, the symmetric jump-diffusion (Merton's model) and the asymmetric jump-diffusions Kou (2002), Synowiec (2008) have taken care of the deficiencies in the Black-Scholes model. However, **most of the existing models are geared towards exact analytical solutions for option pricing and not towards compatibility and suitability with the market price process's behaviour, non-normal and empirical features, and properties of the distributions of the stock market price process**. More so, in the existing jump-diffusion processes as described in equation (1.3), the randomness of the jump term: $J(Q_j)dN_t$, is basically determined by two random processes (which in the actual sense is a compound Poisson process). The first is the random jump times: N_t and the second is the random measure $J(Q_j)$ of jump amplitudes (ie, a measure of the jump sizes). In the existing literature, **the jump-intensity λ is assumed the same for both the upward and downward jump processes. This does not depict the reality of the arrival times of these jumps in the stock price process.** There is, therefore, a need for more suitable measures that can accommodate the above-mentioned features in the dynamics of stock market indices.

This research, therefore, **sought to address these existing gaps by developing suitable jump test models and jump-diffusion models subject to the condition that the distributions of their random jump processes are asymmetric with non-zero location parameters.** These were achieved by the proposed jump test models via the RMPV processes, as well as the **Asymmetric Laplace jump Diffusion,** $(ALJD)$ and the **Modified Double Rayleigh jump-diffusion** $(MDRJD)$ models.

1.3 Research aim and objectives

The aim of this research is to construct Particular Higher-Order Cases (PHOC) jump-estimators, and build suitable models that can accommodate the jumps and non-normal features found in the stock markets indices. The objectives of the work are to:

(1) Obtain jump test models based on particular higher-order cases' asymptotic properties in the realised multipower variation process.

(2) Investigate the presence of jumps via the jump test models of the realised multipower variation process in the stock indices.

(3) Obtain suitable skewed jump-diffusion models with non-zero location parameters for the stock indices.

(4) Derive the probability density functions and the Lévy-Khintchine formula of the log returns for the skewed jump-diffusion models.

(5) Obtain the initial appraisals of the parameters in the model from the stock market data.

(6) Estimate the optimal values of the parameters in the models.

(7) Carry out sensitivity analyses of the varied threshold of jumps on the parameters in the models.

(8) Check for the suitability of the models with the stock market indices data.

1.4 Research methodology

The probability limits and limit in distribution methods were used to obtain the Jump-Test Models (JTM) for the Particular Higher-Order Cases (PHOC) of the RMPV processes. The JTM were used to test for the presence of jumps under the null hypothesis of no jump in the $\Delta(ln\tilde{S}_t)$ via the RCodes in three stock markets, namely: Nigerian, UK, and Japan, comprising, 5334, 2076, and 2076 stock market daily indices observations respectively. The dynamics of the log-returns of the stock price was assumed to be:

$$\Delta(\ln\tilde{S}_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\Delta W_t + J(Q_j^u)\Delta N_t^u + J(Q_j^d)\Delta N_t^d$$

where, $\mu, \sigma, W_t, J(Q_j^u), J(Q_j^d), N_t^u$ and N_t^d are respectively drift and volatility parameters, standard Brownian motion at time t, upward and downward jump measures, and upward and downward finite jump activities, having jump intensities λ_j^u and λ_j^d, respectively. The method of convolution of densities and the Lévy-Itô decomposition method were used to derive the Probability Density Functions (PDF) and the Lévy-Khintchine (LK) formula of two new skewed JD models: Asymmetric Laplace JD (ALJD) and the Modified Double Rayleigh JD (MDRJD) models. The maximum likelihood estimation method was used to estimate the parameters in the models. A sensitivity analysis on the parameters to the varied threshold of jumps was carried out. The Kolmogorov-Smirnov, Anderson-Darlng statistics and the basic moments were used to investigate the suitability of these models to the empirical stock market data, compared with three existing models viz: BS, normal and Double-Exponential JD models.

1.5 Motivation for the study

Most financial data obtained from the market presents jumps in the asset price process; hence, investigating the kind of stochastic process that best models the stock price indices is very important. This will be useful for **proper prediction, risk hedging, and investment decision-making process.**

Most of the existing jump-diffusion models are driven by the existence of their exact analytical solutions, useful in Option Pricing. However, our motivation here is based not only on exact solutions or closed-form solutions but also on the consistency or compliance of models to the market price process.

The *Lévy*-jump diffusion models were considered more suitable for modelling in the case of jumps than the existing diffusion models. This gives the motivation for the study in the thesis. The work is also motivated by the jump test methods in Barndorff-Nielsen *et al.* (2006c) and Aït-Sahalia (2002), as well as the jump-diffusion models given in Merton (1976), Gugole (2016), Yusof and Jaffar(2017), Kou (2002), Kou (2000) and Synowiec (2008).

1.6 Justification of the study

The realised bipower variation process in the **BNS** jump test method has been proven to be a consistent estimator of the integrated variance. In this thesis, this concept was generalised by considering more efficient and suitable estimators of the integrated variance. These estimators are the particular higher-order cases of the realised multipower variation process. They give better results when applied to jump test in the stock indices of finite jumps.

Also, **a generalisation of the existing jump-diffusion models** is needed for better representation of features found in the real market price process.

1.7 Significance of the study

The results of the analyses in this thesis will be of great benefits to **market participants**, since the knowledge of the jump intensities obtained in the models, will depict the measure of risk involved in the stock markets. It is also important to note that **investors** are faced with higher risk in the markets when non compatible models to empirical data obtained from the markets are used for **future prediction of price processes**. Hence, the usefulness of the suitability analyses of the proposed models in this thesis, with the empirical and distributional properties of the real data from the stock market, will be useful **for proper prediction**. The occurrences of jumps in the price process vary from market to market. Hence, the generalised jump-test method provided in this work, is robust and will prove helpful to **financial analysts** in jump-detection.

1.8 Scope of coverage

Stochastic models of stock price processes are grouped into diffusion, jump-diffusion, and pure jump models. These can be further categorised into processes with finite and infinite jump activities. This research focused on jump-diffusion processes whose upward (N_t^u) and downward (N_t^d) jump activities are Poisson processes, assumed to be finite, with jump intensities given as λ_j^u and λ_j^d respectively. Specifically, the processes: $J(Q_j^u)dN_t^u$ and $J(Q_j^d)dN_t^d$ are compound Poisson processes, such that $J(Q_j^u)$ and $J(Q_j^d)$ are non-standard asymmetrically distributed jump amplitude.

The aspect of jump test via the realised multipower variation process is restricted to the condition that the sum of the powers of the variation is equal to two (that is, $\delta(r_1, \ldots, r_m) = 2$).

More so, the stock indices data set tested in this thesis were obtained from three (3) stock markets, namely: the **Nigerian All Share Index**, **UK Stock Market Indices** and **Japan Stock Market Indices**. The three stock markets were selected based on the following reasons: The availability of the stock market price data and the countries' heterogeneous levels in terms of market pricing mechanism, available financial instruments, and sophistication of trading strategies used by agents in the exchanges.

1.9 Organization of the thesis

This thesis is made up of six chapters, which were arranged sequentially from the introduction of the research to the summary and conclusions.

Chapter one introduced the research topic and stated the aim and objectives of the research. This chapter also contained the statement of the problem to be solved. The motivation, justification, significance of the study and the scope of the research, were presented in the chapter.

Chapter two gave a detailed review of the existing models for jump test in discretely-observed price processes. A review of existing jump-diffusion models that were used in literature was also given here. The chapter also highlighted the gap in knowledge and the modifications that were made to the existing models.

Chapter three dealt with some basic definitions, concepts, and a clear description of the BNS jump test method and some existing jump-diffusion models that were applied in this thesis.

In chapter four, the results based on the jump test via the RMPV models with the stock indices data were obtained. The density functions of new jump-diffusion models were derived. The chapter also contained the estimation of the optimal values of the new models' parameters. The suitability of these models was tested with the empirical data sets obtained from the markets and detailed discussions on the obtained results were given.

The discussions of results were presented in chapter five, and the summary, conclusions, contributions to knowledge, and areas of further research were given in chapter six.

CHAPTER TWO

LITERATURE REVIEW

2.1 Preamble

This thesis considered stock indices processes' dynamics when jumps are detected in available data from the stock markets. This is very important since the total variability of a stock price process is not just determined by the *diffusion* process described by the Black-Scholes model in Black and Scholes (1973b), which assumes normal distribution for the stock price, it also depends on the *jump components*, (Trautmann and Beinent, 1994). Jump test methods for financial data have been widely studied in literature, these include both the parametric, (Tauchen and Zhou, 2011) and the non- parametric (Barndorff-Nielsen *et al.*, 2006c and Aït-Sahalia and Jacod, 2009) methods. These methods are restricted to the assumption that assets return follow special cases of semimartingale processes. The above-mentioned helps in financial modelling because they suggest the kind of stochastic models to be applied. Stochastic models with jumps have bridged the gap between the diffusion models (e.g., B-S. models) and the pure jump models, e.g., the VG models, the CGMY models, the NIG models, etc.

Given the above, this chapter reviewed some relevant literature on jump test for stock market data and existing stochastic models when jumps are detected in the stock market data.

2.2 Review of relevant literature

Here, some existing literature where proposed methods for detecting jumps in the face of financial data and stochastic models with jumps were reviewed.

2.2.1 Literature on jump test methods

Jumps are sudden discontinuities that occur spontaneously in the trajectories of price processes as a result of large price movements. In literature, researchers have adopted some methods built on large discrete observations of the price process for each equidistant time interval in a fixed time-space $[0, t]$. For such observations, the asymptotic theories for the variability of the price process have been widely studied.

A vivid observation of a price process $X = \{X_t\}_{t \geq 0}$ defined on a given interval $[0, t]$, such that the observations are made for all discrete times $0 = t_0, t_1, \ldots t_n = t$, where, the j^{th} observed time is given as:

$$j\Delta = j\frac{t}{n} \implies n = \frac{t}{\Delta}, \qquad j = 1, 2, \cdots, n, \tag{2.1}$$

where, n is the number of observations, owing to the recent availability of large financial data, n tends to be so large that Δ's tend to diminish. This depicts a typical real-life intra-day observations of stock prices or inter-day daily data of high volume, which results in high frequency. A considerable number of studies have adopted jump test methods in the discrete case when high-frequency data are available.

Now, owing to the discrete observations described above, a plot of discrete values can be joined together to obtain a full sample path of both the observed values (which could be discrete) and the unobserved values, thereby giving rise to some continuous sample paths. The points of discontinuities in an underlying process could be due to varying discretely-observed values or some valid proofs of the presence of jumps in the process. The question as to whether or not discrete data were generated from a continuous process or possibly discontinuous process was first answered by Aït-Sahalia (2002). The work of Aït-Sahalia (2002) was built on the argument proposed by Karlin and McGregor (1959), where the concept of determining the transition densities of the diffusion process, when it is a markov process was employed. The effect of the **diffusion criteria** on option pricing models was also the focus of the work of Aït-Sahalia (2002). However, it was later observed that the method of Aït-Sahalia (2002) was restricted to the one factor markov property. Therefore, the problem of employing the diffusion criteria when the sample paths of such observations are not markovian in a continuous case was addressed by Carr and Wu (2003).

It is important to note that an underlying price process comprises continuous components, purely discontinuous components, or both. According to Carr and Wu (2003), the rate (speed) at which an underlying price process converges is mainly determined by the two factors: continuous, purely discontinuous components, and the moneyness factors. Their results proved that the convergence rates of **out of the money (OTM)** Option price process were faster, (as the maturity time T tends to zero) in the case of the continuous martingale process as compared to an infinite jump variation process. As an extension of the work of Carr and Wu (2003), a nonparametric approach via the transition density of a discretely sampled jump-diffusion model was proposed by Aït-Sahalia *et al.* (2012).

To determine the dynamics of any process, there is the need to examine and estimate some asymptotic features in discretely-observed price data. In this regard, Andersen *et al.* (2004) and Huang and Tauchen (2005), among others, considered some realised quantities (realised variation, quadratic variation, integrated variance/volatility, and realised power variation) in terms of the effects of jumps on price variability of the process. Limit distribution, which includes the central limit theorem for these quantities were obtained by Jacod (2008), and Barndorff-Nielsen and Shephard (2003b). Lee and Mykland (2007), adopted a nonparametric approach to obtain jump arrival times and realised jump sizes in an underlying price process, given multiple daily observations.

With regards to jump test by wavelet, Xue *et al.* (2014), applied the wavelet methods designed by Wang (1995), Park and Kim (2004, 2006), Fan and Wang (2007) to estimate jump arrival times, based on the asymptotic results of the method. They empirically applied the jump test method by wavelet to the data of the US equity markets. The application of Kernel estimators of continuous sample paths by Wang and Yang (2009), Xia (1998) and Claeskens and Van Keilegom (2004) as well as the Spline method by Koo (1997), was employed by Ma and Yang (2011) in a jump test procedure.

A nonparametric jump test method based on the asymptotic distribution results of the realised power variation for a generalised power p and a Blumenthal-Getoor Index was proposed for jump test by Aït-Sahalia and Jacod (2009) to detect the presence of jumps in high-frequency discrete data. And further to the work of

Aït-Sahalia and Jacod (2009), for finite and infinite number of jumps, since such data come with noise, Aït-Sahalia and Jacod (2011) and Aït-Sahalia (2012) have proffered solutions to accommodate the effects of such a noisy high-frequency data to the test method of Aït-Sahalia and Jacod (2009). Gioa and Vieu (2016) have categorised methods of analysing high-frequency information, and for jump detection.

Pure jump models such as the VG model by Madan and Seneta (1990) and Madan *et al.* (1998), the CGMY model named after **Carr, Geman, Madan and Yor** (Carr *et al.*, 2002), the generalised hyperbolic model of Eberlein and Keller (1995) that has its specification in the Normal Inverse Gaussian (NIG) model, were generously applied to model asset price returns. As a result of the above provision, Jing *et al.* (2012) proposed a method that can be used to check if these available pure jump models were viable enough to fit into real-life situations in the market when data are available in high frequency.

By assuming that the log returns of an asset price process X_t belongs to a class of Brownian semimartingale (BSM) process or Stochastic Volatility semimartingale ($Svsm$) process as described by Jacod and Shiryaev (2003), then, the RPV of such a process which, according to Barndorff-Nielsen and Shephard (2003a) is defined as **the estimated sums of the absolute power of increments** form the basis of the jump-tests that were proposed in Barndorff-Nielsen and Shephard (2003a). It is well known in literature (see Barndorff-Nielsen and Shephard, 2003a) that the realised power variation process is an estimator of the integrated variance, which, in turn, plays a major role in finding the integrated volatility given an $Svsm$ process without jumps. The concept of the jumps with volatility is essential in financial modelling as it dictates the risk measure involved in any given price process. Hence, the works of Merton (1980) and Nelson (1992) have shown that volatility can be obtained by some stipulated methods when the market data is available in a high frequency. The smooth sample paths of the models proposed by the above-named authors were countered by some practical empirical shreds of evidence from the market.

The stock price dynamics studied is synonymous with evaluating and predicting the process of a risky asset since the stock itself is a risky asset. Consequently, the risk is caused by some sudden shocks, which are driven by some unpredictable process. The continuous stochastic models of stock indices are not exempted from these features. In addition to the existing risks in the market, when jumps are introduced into existing models, based on some empirical real-life evidence of these sudden movements, then it results in higher risk in the market, which will later lead to changes in the dynamics of the process.

The presence of jumps in any underlying price process calls for additional parameters in the model; these parameters include risk parameters resulting from jumps, the price intensity of the risk, etc. Bajgrowicz *et al.* (2015) and Calcagnile *et al.* (2018) have *proved that a new inflow of information causes jumps in the market, and these inflows are very unpredictable in themselves.* In addition to the above authors' work, Aït-Sahalia *et al.* (2015) *showed that the presence of jumps in a particular market can spread easily to the other markets across the globe.*

Generally, the importance of the study of jumps in a price process can be categorised into four groups. First, according to Bates (2008), Aït-Sahalia *et al.* (2011) and Liu *et al.* (2018), it was shown that the study of the presence of jumps in a price process could help investors in the market to allocate assets in the right proportion, as well as to make decisions based on optimal portfolio selection since large price movement may result in much market loss. Second, Duffie and Pan (1997) have shown that the study of jumps is relevant to the aspect of risk management. Third, some other studies (Duffie *et al.*, 2000 and Eraker *et al.*, 2003), have it that the presence of jumps could cause market incompleteness. Fourth, the study of jumps will aid in asset pricing when available options cannot be replicated, since some jump risks cannot be erased from the market, according to Duffie and Pan (1997).

Owing to the availability of high-frequency data from the market, there have been some controversial issues (such as heavy noise as a result of high frequency) as regards volatility modelling, as reported by Aït-Sahalia *et al.* (2005) and Andersen *et al.* (2013). The applications of the concept of realised volatility have been treated in the past two decades. Related works include those of Poterba and Summers (1986), Hsieh (1991), Zhou (1996), Taylor and Xu (1997), Andersen *et al.* (2001a), Andersen *et al.* (2001b) and Yaya *et al.* (2018).

In the quest for volatility estimation in the face of high-frequency data, the realised variance converges in probability to the quadratic variation, in a class of continuous Svsm process. In view of this, Barndorff-Nielsen and Shephard (2002a) and Barndorff-Nielsen and Shephard (2003b) obtained the central limit theorem for a special class of stochastic volatility models under some weak conditions subject to the price process. Similarly, estimators of the realised volatility which in turn give estimates of the integrated volatility in the face of jumps were highlighted as the bi-power variation, minimum realised variation, and medium realised variation, respectively, by Barndorff-Nielsen and Shephard (2002b), and Andersen *et al.* (2012). And as such, a comparative analysis of the trio was given in Yaya *et al.* (2018).

The reason for introducing the jump test method via the realised bi-power variation (Barndorff-Nielsen *et al.*, 2006a), is the limitation of the realised power variation process when jumps are added to a continuous Svsm process. In other words, when jumps are imputed into a class of continuous Svsm process, the realised power variation process cannot estimate the integrated variance. Instead, it results to the *sum of integrated volatility and quadratic variation of the imputed jump process, thus showing that the realised power variation is not robust to give an exact estimate of the IV in the face of jumps.* The **BNS-jump test method** named after **Barndorff-Nielsen and Shephard** was therefore introduced for the purpose of determining a tool for a process that proves more robust to jumps; the realised bi-power variation process in Barndorff-Nielsen and Shephard (2001a, 2001b), Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen *et al.* (2006b) and Barndorff-Nielsen and Shephard (2006) is therefore the first order particular case of the realised multipower variation process. in literature, the BNS method has been widely used to test for jumps given discretely-observed financial data of high frequency. More so, Huang and Tauchen (2010) had applied the BNS jump test method alongside Monte-Carlo's analysis to high-frequency data. Their results show that the behaviour of an underlying process changes at a slight touch of a jump. A similar application of the method used in Huang and Tauchen (2010) can also be found in Tauchen and Zhou (2011). More recent applications of the BNS jump test can be found in the works of Xu *et al.* (2016), Gkillas *et al.* (2020a), and Gkillas *et al.* (2020b).

2.2.2 Literature on jump-diffusion models

Many works were carried out with regards to the studies on the existing jump-diffusion models. The most celebrated option pricing model, the B-S model, is found to be deficient in the empirical findings from most financial data. By comparing the features of the normal distribution of the Black-Scholes model to the empirical distribution of log returns of the asset price, it has been observed that the parameter that measures the **asymmetric property (Skewness) in most financial price process gives values that are different from zero, whereas the skewness is zero in the symmetric case of the B-S model**. The skewness's negative value implies a longer tail to the left (the reverse of this implies positive skewness, which shows a long tail to the right than to the left). The empirical skewness of returns of stock indices has been proven to be negatively skewed (asymmetric). In addition to the above, the large movements in the stock price process, which appears more frequently than in the B-S' type of model, results in a case of **excess kurtosis or fat tails** and according to Schoutens (2003), this property is a pointer to the fact that the stock indices price process needs to be represented via jump-diffusion or jump models, as the case may be.

Attempts have been made by many authors to address the above-mentioned lack of compatibility of the B-S model to the empirical distribution of the market data's real-life situation. Under this, Duffie *et al.* (2000) provided an analytical solution in closed form (which is useful for option pricing), a platform to address the **leptokurtic features (excess kurtosis)** found in empirical distribution. This is very necessary since this feature subject to risk-neutral measure results in "volatility smile" in option prices (see Synowiec, 2008). The proposed analytical platform gave birth to an Affine jump-diffusion (AJD) model where the distribution of the jumps is a Poisson process. When the distribution of the jump sizes in the AJD model takes a normal path, then the AJD model becomes a normal jump-diffusion, as can be found in the work of Hanson and Westman (2002). Their work showed that the jumps in the model were able to track the S and P 500 stock index data in terms of capturing **large jumps, extreme outliers, the problem of negative skewness, and leptokurtic features**. A few authors have applied the work of Duffie *et al.* (2000) to different fields. These include modelling and pricing electricity derivatives for evaluating jump activities (Culot *et al.*, 2006, Nomikos and Soldatos, 2008), as well as jump volatility in the commodity market (Da Fonseca and Ignatieva, 2019), and stochastic volatility models of the AJD (Belaygorod and Zardetto, 2013). The work of Glasserman and Kim (2009) extends the closed form of the AJD model to a practicable technique. Broadie and Kaya (2006) introduced a method for finding the exact simulation of the market price using the AJD model.

A simplified form of the normal jump-diffusion of Hanson and Westman (2002) is the Merton model by Merton (1976) which was used in Gugole (2016) where, it was proved to have performed better than the famous Black-Scholes model, under a comparative analysis. The Merton jump-diffusion model is one of the first improvements of the B- S model. It is a special kind of jump-diffusion that has many similarities with the B-S model, except for the jump term, which is a compound Poisson process comprising two random variables, namely: the jump intensity (average number of jumps) which is a Poisson process, and the jump size of the log-stock price which obeys normal distribution. Note that the jump intensity and the log stock price's jump size are assumed to be independent random variables in the Merton model. Since Merton's proposed work, there have been numerous applications of the model to diverse fields in financial modelling. These include the works of Black and Cox (1976), Geske (1977), Zhou (1997) and Zhou (2001), which showed that the Merton model can be used to obtain the dynamics of a firm's value. The probability of companies' and firms' defaults has been used in the Merton model by Yusof and Jaffar (2017) and Vestbekken and Engebretsen (2016).

Apart from the Merton model, one of the extensions to the B-S model is the Kou model in Kou (2000) and Kou (2002). This model, just like the Merton model, addresses the leptokurtic features and the non-constant volatility of the returns of stock prices. The measure of the jump size in the Merton model, which follows a normal distribution, is replaced by a double exponential distribution to cater for fat tails and high peaks found in empirical distributions. To accommodate these empirical features, it was found in Kou (2002) that the double exponential distribution possesses **a high peak** and **heavy tail** parameters. The work of Sezgin-Alp (2016) proved that Kou's model performs better than the B-S model in the Turkish stock market, and the fuzzy version of Kou's model was presented by Zhang *et al.* (2012). A stochastic volatility model of interest rate was merged with Kou's model to achieve an option pricing model in Chen *et al.* (2017). Applications of the Kou model in literature include the pricing of a compound option. Real option estimation via the double exponential jump-diffusion process has been reported by Liu *et al.* (2018) and Hillman *et al.* (2018).

A replacement of the double exponential distribution in the Kou model is the uniform distribution. The Uniform jump-diffusion (UJD) model was proposed by Hanson *et al.* (2004b) because the normal jump-diffusion and double exponential jump-diffusion models have just a small deviate from the diffusion models in the sense of their distributional peak. The UJD model was used in place of the existing models to account for possible negative and positive jumps. Subsequently, a stochastic volatility jump-diffusion by Yan and Hanson (2006) was introduced to merge a particular stochastic volatility model with the UJD model. Owing to most jump-diffusions' intractable nature, Monte Carlo's approach was adopted by Zhu and Hanson (2005) as a tool for computing European option prices for a log uniform jump-diffusion model. Ahlip and Prodan (2015) derived an analytical result for the European call option following the format of the work of Zhu and Hanson (2005). To obtain more suitable models that can fit the real market better than the B-S models, so many works have been done in the sense of additional parameters to the existing models. In line with the above-mentioned, the hyper-exponential jump-diffusion model by Cai *et al.* (2009) and Crosby *et al.* (2010), the double Rayleigh jump-diffusion model and the double normal jump-diffusion model by Synowiec (2008) were added to the group of jump-diffusion models. The multiple parameter models of the jump-diffusion process were proposed by Xu *et al.* (2016), namely: the BS-SGT and the Kou-SGT models, which were obtained from the existing B-S and the double exponential jump-diffusion models.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Preamble

In this chapter, some basic definitions, concepts and Mathematical descriptions of some existing jump diffusion models to be applied in this thesis were presented.

Study one contains some basic definitions in Financial Mathematics, that serves as the bedrock of stochastic calculus. The basic concepts of the *Lévy* process and its properties, martingales and semimartingales, stochastic volatility semimartingales were given in study two. The realised power variation, the realised variance, the realised multipower variation process as well as their asymptotic properties, which forms the basis for the jump-test method used in this thesis, were contained in study three. In study four, a breakdown of some existing models of the jump-diffusion process found in literature, stochastic formulas for solving the dynamics of stochastic differential equation with and without jumps were presented.

3.2 STUDY ONE

Introduction

In this study, some basic definitions in financial mathematics in relation to concepts and models in stochastic calculus were presented. These form the bedrock for the methods used in this thesis. These definitions, concepts and models were given according to the following authors: Shreve (2004), Schoutens (2003), Kyprianou (2006), Ugbebor (2009), Durrett (2019), Schoutens and Cariboni (2010), Ekhaguere (2009) and Protter (2005).

3.2.1 Basic Definitions

Definition 3.1: A measurable space. (Ugbebor, 2009, Durrett, 2019)
Given a set Ω, collection of subsets $\mathcal{F}$, which forms a σ-algebra, satisfying the following conditions:
1. $\Omega \in \mathcal{F}$
2. $E \in \mathcal{F}(\implies E^c \in \mathcal{F})$
3. Let $E_i; i = 1, 2, ..., n$ be such that $E \in \mathcal{F}$ and, $\bigcup_{i=1}^{n} E_i \in \mathcal{F}$. Then, the pair $(\Omega, \mathcal{F})$ is called a **measurable space.**

Definition 3.2: A set function (Ugbebor, 2009, Durrett, 2019)
A **set function**, Ψ is defined on a non-empty class $\mathcal{C}$ of sets in a space Ω by assigning to each set $E \in \mathcal{C}$ a single number, $\Psi(E)$, (finite or infinite) known as the value of Ψ at E

Remark 3.1 (Durrett, 2019)
Suppose $E_1, E_2 \in \mathcal{C}$, such that: $\Psi(E_1) = +\infty$ and $\Psi(E_2) = -\infty$

Then,

$$\Psi(\Omega) = \Psi(E_1) + \Psi(E_1) = +\infty \tag{3.1}$$

and

$$\Psi(\Omega) = \Psi(E_2) + \Psi(E_2) = -\infty \tag{3.2}$$

From equations (3.1) and (3.2) above, the value of $\Psi(\Omega) \neq \Psi(\Omega)$ since Ψ is a single-valued function, which is a contradiction.

Definition 3.3: A additive set function (Ugbebor, 2009, Durrett, 2019)

A set function is said to be **additive** if for all sets E_1, E_2 such that $E_1 \cap E_2 = \phi$, $\Psi(E_1 \cup E_2) = \Psi(E_1) + \Psi(E_2)$ and by finite additivity,

$$\Psi\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} \Psi(E_i) \tag{3.3}$$

where, $E_i \cap E_j = \phi$ for all $i \neq j$

This is called the finite additivity property of a set function.

An **additive set function** Ψ is a set of function which has the additivity property, and in addition, one of the values $+\infty, -\infty$ is NOT allowed. To fix ideas, $\Psi \neq -\infty$

1. If all the values of Ψ are finite, then Ψ is said to be finite written as: $|\Psi| \leq \infty$

2. If every set in the given class $\mathcal{C}$ at which Ψ is a countable unions of sets in $\mathcal{C}$, which Ψ is finite, then Ψ is said to be σ-finite.

Definition 3.5: A content (Ugbebor, 2009, Durrett, 2019)

A positive set function which is finitely positive is called a **content**.

Definition 3.6: A measure and positive measure (Ugbebor, 2009, Durrett 2019)

Given a countable class of disjoint sets $E_1, E_2, \cdots E_n$, if $\Psi\Big(\bigcup_{n=1}^{N} E_n\Big) = \sum_{n=1}^{N} \Psi(E_n)$, then Ψ is said to be a **countably positive set function** (σ-positive set function). A set function which is σ-positive is called a **measure**. A **positive measure** is a set function μ which is defined on a σ-algebra $\mathcal{F}$ given as:

$$\mu\colon \mathcal{F} \to [0, \infty] \tag{3.4}$$

with range in $[0, \infty]$ and is countably $\sigma-$ positive.

Definition 3.7: Continuity of a positive set function (Ugbebor, 2009)

A positive set function Ψ is said to be

1. continuous from below if $\Psi(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \Psi(E_n)$ for every increasing sequences $\{E_n\} : E_1 \subset E_2 \subset E_3 \subset \cdots$

2. continuous from above if $\Psi(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \Psi(E_n)$ for every decreasing sequence $\{E_n\} : E_1 \supset E_2 \supset E_3 \supset \cdots$ and $\Psi(E_n) \leq \infty$ for some values of $n = n_0$

Remark 3.2: (Ugbebor, 2009)

The following remarks on the continuity of the positive set function are highlighted:

1. continuity at ϕ reduces to continuity from above.

2. continuity from above, the condition that $\Psi(E_n) \leq \infty$ for some values of $n = n_0$ is necessary.

3. A set function, Ψ is continuous, if it is continuous from above and continuous from below.

Theorem 3.1: (Ugbebor, 2009)

Every σ-additive set function is finitely positive.

Theorem 3.2: (Ugbebor, 2009)

Let Ψ be finitely positive, finite and continuous at the empty set ϕ, then Ψ is σ-additive.

Definition 3.8: A measure space (Ugbebor, 2009, Durrett, 2019)

Given a measure μ, which is a positive set function defined on $(\Omega, \mathcal{F})$ and given as:

$$\mu \colon \mathcal{F} \to \mathbb{R} \tag{3.5}$$

satisfying the following conditions:

1. $\mu(\phi) = 0$

2. whenever $E_1, E_2, E_3 \cdots$ is a class of pairwise sets such that,

$$\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n) \tag{3.6}$$

Then, the triple $(\Omega, \mathcal{F}, \mu)$ is called **a measure space.**

Definition 3.9: Probability measure and probability space (Durrett, 2019)

A measure $\mathbb{P}$ such that $\mathbb{P}(\Omega) = 1$ is called **a probability measure.** The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space** where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, a subset of Ω.

Definition 3.10: A Filtration (Ekhaguere, 2009)

A non-decreasing family $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ of sub $\sigma-$algebra of $\mathcal{F} : \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F}$ for all $0 \leq s \leq t \leq T$; where $\mathcal{F}_t$ stands for information available at time t then $\mathbb{F} = \{\mathcal{F}_t : 0 \leq t \leq T\}$ is called **a filtration** which represents information flow.

Definition 3.11: A Filtered probability space (Ekhaguere, 2009)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$ satisfying the following conditions:

1. $\mathcal{F}_0$ contains all the $\mathbb{P}$-null members of $\mathcal{F}$, that is:

$$\mathbb{P}(E) = 0 \implies \mathrm{E} \in \mathcal{F} \tag{3.7}$$

2. $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $0 \leq s \leq t$

3. $\mathbb{F}$ is right continuous in the sense that

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s \tag{3.8}$$

Then, $\mathbb{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$ is called a filtration of $\mathcal{F}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is called a **Filtered probability space** or a **Stochastic basis**.

Definition 3.12: A random variable (Ugbebor, 2009)

Given a sample space Ω, a real-valued function defined on the sample space Ω, such that

$$\tilde{X} : \Omega \to \mathbb{R} \tag{3.9}$$

is called a **random variable**, and $\tilde{X}$ may be discrete or continuous.

Definition 3.13: Distribution function (Durrett, 2019)

The **distribution function** of a random variable $\tilde{X}$ is given by $F(x) = P(\tilde{X} \leq x)$, where, $dF(x) = f(x)dx$ and $f(x)$ is the **density function** of $\tilde{X}$

Remark 3.3 (Ugbebor, 2009)

A random variable $\tilde{X} : \Omega \to \mathbb{R}$ is said to be integrable if:

$$\int_{\Omega} \Big|X(\omega)\Big| d\mathrm{F}(\omega) = \int_{\mathbb{R}} |x| f(x)dx < \infty \tag{3.10}$$

where, $\omega \in \Omega$

Definition 3.14: Mathematical expectation/Variance of $\tilde{X}$. (Durrett, 2019)

Let $\tilde{X} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then the **mathematical expectation** of $\tilde{X}$ is defined as:

$$\mathbb{E}(\tilde{X}) = \int_{\Omega} \tilde{X}(\omega) \mathbb{P} d\omega) = \int_{\mathbb{R}} \tilde{X} f(x)dx \tag{3.11}$$

If $\tilde{X} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, then the **Variance** of $\tilde{X}$ is defined as:

$$Var(\tilde{X}) = \mathbb{E}(\tilde{X} - \mathbb{E}(\tilde{X}))^2 \tag{3.12}$$

Definition 3.15: Conditional expectation (Durrett, 2019)
Let $\tilde{X}$ be a random variable with $\mathbb{E}(|\tilde{X}|) < \infty$, then for $\tilde{X} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}_s$ a sub σ-algebra of $\mathcal{F}$, the random variable $E(\tilde{X}|\mathcal{F}_s)$ is called the **Conditional expectation** of $\tilde{X}$ given $\mathcal{F}_s$

Remark 3.4: Properties of Conditional expectation (Durrett, 2019)
Let $\alpha, \beta \in \mathbb{R}$, $\tilde{X}, \tilde{Y}, S_n, n = 1, 2, \cdots$, be $\mathbb{R}$-valued members of $L^1(\Omega, \mathcal{F}, \mathbb{P})$; and $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be sub σ-algebra of $\mathcal{F}$, then the following is true:
1. The conditional expectation is linear:

$$E((\alpha\tilde{X} + \beta)|\mathcal{F}) = \alpha E(\tilde{X}|\mathcal{F}) + \beta E(\tilde{X}|\mathcal{F})$$

almost surely.
2. The conditional expectation is positive preserving i.e, for a nonnegative random variable $\tilde{X}$,

$$E(\tilde{X}|\mathcal{F}) \geq 0$$

3. $E(1|\mathcal{F}) = 1$, in general, $E(c|\mathcal{F}) = c$, where c is a constant.
(a) If X is $\mathcal{F}$-measurable, then $E(\tilde{X}|\mathcal{F}) = \tilde{X}$ almost surely.
(b) If $\tilde{X}\tilde{Y} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\tilde{X}$ is $\mathcal{F}$-measurable, then

$$E(\tilde{X}\tilde{Y}|\mathcal{F}) = \tilde{X}E(\tilde{Y}|\mathcal{F})$$

4. Tower Property
If $\mathcal{F}_1 \subset \mathcal{F}_2$, then,

$$E(E(\tilde{X}|\mathcal{F}_2)|\mathcal{F}_1) = E(\tilde{X}|\mathcal{F}_1)$$

Similarly,

$$E(E(\tilde{X}|\mathcal{F}_1)|\mathcal{F}_2) = E(\tilde{X}|\mathcal{F}_1)$$

almost surely.
5. If $S \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $S_n \to S$ in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, then

$$E(S_n|\mathcal{F}) \to E(S|\mathcal{F}) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$$

6. Jensen's inequality
If $f : \mathbb{R} \to \mathbb{R}$ is convex and $f(x) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then

$$f(E(\tilde{X}|\mathcal{F})) \leq E(f(\tilde{X})|\mathcal{F})$$

In particular, since $x \to |x|^p$, $1 \leq p \leq \infty$, $x \in \mathbb{R}$ is convex, then,

$$||E(X|\mathcal{F})||_p \leq ||X||_p$$

almost surely $\tilde{X} \in L^p(\Omega, \mathcal{F}, \mathbb{P})$

7. The random variable $\tilde{X}$ is independent of $\mathcal{F}$ iff for every measurable function

$$g\colon \mathbb{R} \to \mathbb{R}$$

such that $g(\tilde{X}) \in L^1(\Omega, \mathcal{F}, \mathbb{F})$

$$\mathrm{E}(g(\tilde{X})|\mathcal{F}) = \mathrm{E}(g(\tilde{X}))$$

Definition 3.16: Characteristic function and exponent (Schoutens, 2003)

Let $\tilde{X}$ be a random variable whose distribution function is given as:

$$\mathrm{F}(x) = \mathbb{P}(X \leq x)$$

The **Characteristic function** $\phi_X(u)$ of the random variable X is the Fourier-Stieltjes transform of the distribution.

Thus, $\phi_{\tilde{X}}(u)$ is defined as:

$$\phi_{\tilde{X}}(u) = \mathbb{E}\Big(exp(iu\tilde{X})\Big) = \int_{\mathbb{R}} exp(iu\tilde{X})d\mathrm{F}(x) \tag{3.13}$$

where, i is the imaginary number ($i = -1$).

The **characteristic exponent** sometimes called the **cumulant characteristics function**.

$$\psi(u) = log\mathbb{E}\Big(exp(iu\tilde{X})\Big) = log\phi(u) \tag{3.14}$$

$$\implies \quad \phi_{\tilde{X}}(u) = \exp\Big(\psi(u)\Big)$$

Definition 3.17: Stochastic process (Schoutens, 2003, Ugbebor, 2009)

A **Stochastic process**, $X = \{X_t : 0 \leq t \leq T\}$ is a family of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by a set $\mathrm{T} \subset \mathbb{R}$ such that:

$$X : T \to \mathbb{R}^d \tag{3.15}$$

$$t \mapsto X(t)$$

i.e

$$X(t)\colon \Omega \to \mathbb{R}^d$$

$$: \omega \mapsto X(t)(\omega) = X(t, \omega)$$

Remark 3.5 (Schoutens, 2003)

The following are remarked on the stochastic process:

1. The process $X_t(\omega)$ is said to be **measurable** if for all $\omega \in \Omega,\ X(t, \omega) : \Omega \to \mathbb{R}$ is a measurable function.

2. The process $X_t(\omega)$ is **adapted** to the filtration $\mathbb{F}$, or simply **$\mathbb{F}$-adapted**, if X_t is F_t-measurable ($X_t \in \mathcal{F}_t$), this implies that the value of X_t is unknown at time t.

3. The process $X_t(\omega)$ is $\mathbb{F}$ **- predictable** if $X_t \in \mathcal{F}_t = \bigcup_{s<t} \mathcal{F}_s$ which implies that X_t is $\mathcal{F}_t$-measurable that is the value of X_t is strictly known before time t.

4. The process $X_t(\omega)$ is **Gaussian** if for all $t_1 < t_2 < \cdots < t_k$ the k-dimensional random variables are normally distributed.

5. The process $X_t(\omega)$ is a **Markov process** if for all $t_1 < t_2 < \cdots < t_k$,

$$\mathrm{P}\Big(X_{t_k} \leq x_{t_k} | X_{t_{k-1}}, \cdots x_{t_1}\Big) = \mathrm{P}\Big(X_{t_k} \leq x_k | X_{t_{k-1}}\Big) \tag{3.16}$$

6. The process $X_t(\omega)$ is said to be **integrable** if:

$$\int_{\Omega} |X_t(\omega)| d\mathbb{P}(\omega) = \int_{\mathbb{R}} |x| \mathbb{P}(x) dx \ < \infty \tag{3.17}$$

Definition 3.18: A continuous stochastic process (Kyprianou, 2006)

A stochastic process $X_t(\omega)$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is **stochastically continuous** or **continuous in probability** if for every $\epsilon > 0$:

$$\lim_{s \to t} \mathbb{P}\big(|X_s(\omega) - X_t(\omega)| > \epsilon\big) = 0 \tag{3.18}$$

Definition 3.19: Independent increment of a stochastic process (Kyprianou, 2006)

The increments of a stochastic process $X_t(\omega)$ are called independent if increments $X_{t_1} - X_{t_2}$ and $X_{t_3} - X_{t_4}$ are independent random variables, whenever the two time intervals $t_1 \leq t \leq t_2$ and $t_3 \leq t \leq t_4$ do not overlap.

Definition 3.20: Stationary increments of a stochastic process (Kyprianou, 2006)

The increments of a stochastic process $X_t(\omega)$ are called **stationary**, if the probability distribution of any increment $X_s - X_t$ depends only on the length $s - t$ of the time interval, i.e increments with equal intervals are identically distributed.

Definition 3.21: A *cádlág* process (Kyprianou, 2006)

A stochastic process $X_t(\omega)$ is called a *cádlág* process if it is right continuous with left limit. That is,

$$X(t_-) = \lim_{\substack{s \to t \\ s < t}} X(s); \qquad X(t_+) = \lim_{\substack{s \to t \\ s > t}} X(s) \tag{3.19}$$

and

$$X(t_-) = X(t_+)$$

for $t \in [0, T]$

Definition 3.22: Jump process (Kyprianou, 2006)

The jump process ΔX_{t_i} of a stochastic process X_t with right continuity and left limits is given as:

$$\Delta X_{t_i} = X_{t_i} - X_{t_{i-1}} \tag{3.20}$$

where, t_i is the jump time of the process for $i = 1, 2, \ldots, n$

Definition 3.23: Jump size (Kyprianou, 2006)

The jump size of a stochastic process X_t is given as:

$$\Delta X = X(t_+) - X(t_-),\ t > 0 \tag{3.21}$$

Definition 3.24: A martingale (Schoutens, 2003, Durrett, 2019)

A **stochastic process**, $X = X_t : 0 \leq t \leq T$ is a **Martingale** defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, if the following conditions are satisfied

1. X is $\mathbb{F}$-adapted.
2. $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$.
3. $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$, almost surely for $0 \leq s \leq t$

Remark 3.6 (Durrett, 2019)

In the above, the stochastic process $X = \{X_t : t \geq 0\}$ is a submartingale and supermartingale if the third condition given above is respectively, $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$ and $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$

Definition 3.25: Brownian motion (Kyprianou, 2006)

A stochastic process $W = \{W_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Brownian motion**, if, the probability that the process starts at zero is one that is, $\mathbb{P}(W_0 = 0) = 1$; it possesses stochastically independent increments such that $W_t - W_s$ is independent of $W_u - W_v$ for $0 \leq v \leq u \leq s \leq t$; the increment: $W_t - W_s$ is normally distributed with mean zero and variance $(t - s)$ and both $W_t - W_s$ and W_{t-s} have the same distribution such that $\mathbb{E}[W_t - W_s] = \mathbb{E}[W_{t-s}]$.

Definition 3.26: Geometric Brownian Motion (Kyprianou, 2006)
Let $\{W_t : \ t \geq 0\}$ be a Brownian motion, then a stochastic process $\{X_t : \ t \geq 0\}$ satisfying the Stochastic Differential Equation (SDE):

$$dX_t = X_t(\hat{\mu} dt + \sigma dW_t), \ t \geq 0 \tag{3.22}$$

where, $X_{t_0} = x_0$, $\hat{\mu} \in \mathbb{R}$ and $\sigma \geq 0$ is called a **Geometric Brownian Motion**.

3.3 STUDY TWO

Basic concepts of the *Lévy*-process and semimartingales

In this study, the basic concepts of the *Lévy*-process, its properties which include the Lévy-Khintchine formula and some very important results are discussed. The concepts of semimartingales in its continuous form ($Svsm^c$) and discontinuous form ($Svsm^j$) are also highlighted.

3.3.1 Basic concepts of the *Lévy* process

The definition of the *Lévy* process, its properties, examples and some important results of the *Lévy* process were given below.

Definition 3.27: A *Lévy* process
A stochastic process $X = \{X_t : t \geq 0\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $X_0 = 0$ is called a *Lévy* **process** if it possesses: **independent increments**, **Stationary increments**, **Stochastic Continuity** and the *Cádlág* properties as defined in definition (3.17) - (3.20) above.

3.3.1.1 Properties of a *Lévy* process

Here, some very important properties of the *Lévy* process, which include the infinitely divisibility of a *Lévy* process, the *Lévy*-Khintchine formula and *Lévy-Itô* decomposition are discussed. In the sequel, the definition of infinitely divisible distribution is given.

Definition 3.28: Infinitely divisible distribution (Sato, 1999)
Suppose $\phi(u)$ is the characteristic function of a random variable X. If for every positive integer n, $\phi(u)$ is also the n^{th} power of the characteristics function, then, the distribution is **infinitely divisible**. That is, in terms of X, it means that one could write for any n:

$$X = Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_n^{(n)} \tag{3.23}$$

where, $Y_i^{(n)}, i = 1, 2, \ldots, n$ are independent and identically distributed $(i.i.d)$ random variables all following a law with characteristics function $(\phi(z))^{\frac{1}{n}}$.

Example 3.1

The normal distribution $\big(\tilde{X} \sim \mathrm{N}(\mu, \sigma^2)\big)$ is infinitely divisible. That is,

$$\phi_\mu\Big(u; \mu, \sigma^2\Big) = \Big(e^{iu\mu} \cdot e^{(-\frac{1}{2n}\sigma^2 u^2)}\Big)^n \tag{3.24}$$

$$= \Big(\phi_n(u)\Big)^n$$

where, $\phi_n(u)$ is the characteristic function of $Y \sim N(\frac{\mu}{n}, \frac{\sigma^2}{n})$, given that $Y_i^{(n)},\ i = 1, \ldots, n$ are independent and identically distributed random variables and,

$$X = Y_1^{(n)} + \cdots + Y_n^{(n)} \sim N(\mu, \sigma^2) \tag{3.25}$$

Remark 3.7 (Sato, 1999)

The next theorem gives a close relationship between the distribution of the *Lévy* process at time t and the concept of infinitely divisible distribution as shown by De Finetti (1992), (see Sato, 1999, for proof).

Theorem 3.3: Infinitely divisible distribution of a *Lévy* **process**. (Sato, 1999, Schoutens, 2003)

Let $X = \{X_t : t \geq 0\}$ be a *Lévy* process, then X has an infinitely divisible distribution F for every t. Conversely, if F is an infinitely divisible distribution, then there exist a *Lévy* process X such that the distribution of X is given by F.

Theorem 3.4: *Lévy***-Khintchine formula for** *Lévy* **process.** (Sato, 1999, Applebaum, 2009)

Suppose $\mu \in \mathbb{R},\ \sigma \geq 0$ *and* $\nu(dx)$ is a measure concentrated on $\mathbb{R}\backslash\{0\}$ such that,

$$\int_{\mathbb{R}} (1 \wedge x^2)\nu(dx) < \infty \tag{3.26}$$

where, the *Lévy* triple $\Big(\mu, \sigma^2, \nu(dx)\Big)$ is defined for $\mu \in \mathbb{R}$. Then, there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a *Lévy* process is defined having a characteristic exponent $\psi(u)$, where,

$$\psi(u) = i\mu u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \Big(1 - e^{iux} - iux 1_{\{|x|<1\}}\Big)\nu(dx) \tag{3.27}$$

for $\psi(u)$ as given in definition (3.16) and $1_{\{|x|<1\}}$ is an indicator function with:

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge x^2)\nu(dx) < \infty \tag{3.28}$$

Equation (3.27) is called the *Lévy*-Khintchine formula which gives an expression for the characteristics exponent $\psi(u)$ of a *Lévy* process.

Remark 3.8 (Sato, 1999)

From the above, *Lévy* process can be decomposed into three independent components:

1. a deterministic drift with rate μ.

2. a continuous path diffusion volatility σ^2

3. a jump process with *Lévy* measure $\nu(dx)$

Thus, the triple $\Big(\mu, \sigma^2, \nu(dx)\Big)$ is referred to as *Lévy* **triple**.

Theorem 3.5: *Lévy-Itô* **Decomposition.** (Sato, 1999)

Given a *Lévy*-triple $\Big(\mu, \sigma^2, \nu(dx)\Big)$ where $\mu \in \mathbb{R}, \sigma > 0$ *and* $\nu(dx)$ is a measure satisfying $(\{0\}) = 0$ *and* $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. Then, there exist a probability space $\Big(\Omega, \mathcal{F}, \mathbb{P}\Big)$ on which four independent *Lévy* process exist, $X^{(1)}, X^{(2)}, X^{(3)},$ *and* $X^{(4)}$ where $X^{(1)}$ is a constant drift, $X^{(2)}$ is a Brownian motion, $X^{(3)}$ is a Compound Poisson process and $X^{(4)}$ is a square integrable (pure jump) martingale with almost surely countable number of jumps of magnitude less than 1 on each finite time interval. Taking $X = X^{(1)} + X^{(2)} + X^{(3)} + X^{(4)}$, then, there exist a probability space on which a *Lévy* process $X = \{X_t, t \geq 0\}$ with characteristics exponent.

$$\psi(u) = i\mu u + \frac{i\sigma^2 u^2}{2} + \int_{\mathbb{R}} \Big((e^{iux} - 1 - iux)1_{\{|x|<1\}}\Big)\nu(dx) \tag{3.29}$$

for all $u \in \mathbb{R}$ is defined.

3.3.1.2 Examples of *Lévy* process

In this subsection, some examples of *Lévy* processes with regards to their density functions were given.

Example 3.2 The Brownian Motion Process. (Applebaum, 2009, Schoutens and Cariboni, 2010)

The Brownian motion process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Lévy* process $W = \{W_t, t \geq o\}$ satisfying the following conditions:

1. $W_t \sim N(0, t)$ for each $t \geq 0$

2. $W_t = \{W_t, t \geq o\}$ has independent increment

3. W_t has continuous sample path

4. $W = \{W_t, t \geq o\}$ has stationary increments

It is important to note that the most applied process in financial modelling is the **Geometric Brownian Motion** formed from the **Brownian Motion** (Schoutens and Cariboni, 2010). The famous Black-Scholes model for stock price dynamics in continuous time is obtained from this process:
Thus, a stochastic process $\mathrm{S} = \{\mathrm{S}_t, t \geq 0\}$ is a geometric Brownian Motion if it satisfies the stochastic differential equation.

$$d\mathrm{S}(t) = \mu \mathrm{S}(t)dt + \sigma \mathrm{S}(t)dW(t), \ \mathrm{S}_0 > 0 \tag{3.30}$$

where, $W = \{W_t, t \geq 0\}$ is a standard Brownian motion, $\mu \ and \ \sigma$ are respectively the drift volatility parameters.

Example 3.3: The Poisson Process. (Applebaum, 2009)
The Poisson process N_t of intensity $\lambda > 0$ is a *Lévy* process which is defined on the positive integers $\left(\mathrm{N} \bigcup \{0\}\right)$ given that each $\mathrm{N}_t \sim \pi(\lambda t)$ and it's probability density at the point $\mathrm{N}(t) = n$ is equal to:

$$\mathbb{P}\Big(\mathrm{N}(t) = n\Big) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \ n = 0, 1, 2, \ldots \tag{3.31}$$

A Poisson process with intensity $\lambda > 0$ satisfies the following conditions:
1. The paths of N_t are $\mathbb{P}$-almost surely right continuous, left limits.
2. $\mathbb{P}(N_0 = 0) = 1$.
3. For $0 \leq s \leq t, \ \ N_t - N_s$ is equal in distribution to $N_t - s$.
4. For $0 \leq s \leq t, \ \ N_t - N_s$ is independent of $\{N_u : u \leq s\}$
5. For each $t > 0, \ N_t$ is equal in distribution to a poisson random variable with parameter λ.

3.3.2 Concepts of stochastic integrals and semimartingales

The concepts of stochastic integrals and semimartingales in this subsection, were presented.

Definition 3.29: Finite Variation Process. (Jacod and Shiryaev, 2013)
A stochastic process defined as $X = X_t : 0 \leq t \leq T$ is said to be of finite variation on $\mathbb{R}_+$ if for each $t \in \mathbb{R}_+$, then there exist a finite constant k_+ such that:

$$\sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right| \leq k_+ \tag{3.32}$$

for all finite partition $0 = t_0 < t_1 < \cdots < t_n = t$ of $[0, t]$.

Definition 3.30: Stopping Time. (Protter, 1992)

Given an adapted, *Cádlág* Stochastic process

$$X : \Omega \to \mathbb{R}_+ \tag{3.33}$$

and the filtration F_t. Then the random variable

$$\tau(w) = inf\{t > 0 : X_t(w) \in \mathcal{F}_t\} \tag{3.34}$$

is called a **Stopping time**.

Definition 3.31: Stopping time $\sigma-$algebra (Protter, 2005)

Let $\tau(w)$ be a stopping time, the stopping time $\sigma-$algebra $\mathcal{F}_\tau$ is defined as:

$$\{\beta \in \mathcal{F} : \beta \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall\ t \geq 0\}$$

Definition 3.32: A Square Integrable Martingale Let X be a martingale as given in Definition (3.24) in this thesis with $X_0 = 0$ and $\mathbb{E}(|X_t^2|) < \infty$, for each $t > 0$ then, X_t is called a **Square Integrable Martingale**.

Definition 3.33: Uniformly Integrable Martingale. (Protter, 2005)

Let $(X_n)_{n \in A}$ be a family of some random variables, it is said to be **uniformly integrable** if:

$$\lim_{n \to \infty} \left(\sup_n \int_{|X_n| \geq N} |X_n| d\mathbb{P} \right) = 0 \tag{3.35}$$

Theorem 3.6 (pg. 9, Protter, 2005)

Given that X_t is a right continuous martingale, then, $\{X_t\}_{t \geq 0}$ is said to be uniformly integrable if and only if $Y = \lim_{t \in \infty} X_t$ exist almost surely where $\mathrm{E}(|Y|) < \infty$ *and* $\{X_t\}_{t \geq 0}$ is a Martingale.

Definition 3.34: Local martingales (Protter, 2005)

Let $\tau_1 < \tau_2 < \cdots$ be a sequence of increasing stopping times satisfying

$$\lim_{n \to \infty} \tau_n = +\infty \ \textit{almost surely}$$

and

$$X_t \wedge \tau_n 1_{\{\tau_n > 0\}}$$

(where $t \wedge \tau_n$ is $min(t, \tau_n)$) which is also a stopping time) is a uniform integrable Martingale for all n. Then the adapted *Cádlág* stochastic process $\{X_t\}_{t\geq 0}$ is called a **local Martingale**.

Definition 3.35: A simple predictable process (Lamberton and Lapeyre, 2011)
A process $(P_t)_{t\geq 0}$ is called a simple predictable process if it can be expressed as:

$$P_t = P_0 1_{\{0\}}(t) + \sum_{i=1}^{n} P_i \ 1_{(\tau_i, \tau_{i+1}]}(t) \tag{3.36}$$

where, $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n+1} < \infty$ is a finite sequence of stopping times and P_i is $\mathcal{F}_{t_{i-}}$ measurable and bounded $\Big(P_i \in \mathcal{F}_{t_{i-1}}$ and $\mathbb{E}(|P_i|) < \infty\Big)$ almost surely, $0 \leq i \leq n$

Remark 3.9 (Protter, 2005)
Note that in the above definition, $\tau_1 = \tau_0 = 0$ implies that there is no difference between the stopping times τ_0 and τ_1. Equation (3.36) can also be expressed as:

$$\sum_{i=1}^{n} P_i 1_{(\tau_i, \tau_{i+1}]}(t) = \sum_{i=1}^{n} P_i \Big(1_{\tau_{i+1}}(t) - 1_{\tau_i}(t) \Big) \tag{3.37}$$

Definition 3.36: Stochastic integral of a simple predictable process (Lamberton and Lapeyre, 2011)
The stochastic integral of a simple predictable process $\{P_t\}_{t\geq 0}$ is given as the continuous process $I\{P_t\}_{t\geq 0}$ defined for any $\tau \in (\tau_k, \tau_{k+1}]$.

$$I_W(P)_t = \int_0^t P_s d\mathrm{W}_s \tag{3.38}$$

Definition 3.37: A Total Semimartingale (Jacod and Shiryaev, 2013)
A process $\{X_t\}$ is called a **Total Semimartingale** if X is a *Cádlág*, adapted and $I_W(X_t)$ is continuous

Definition 3.38: A Classical Semimartingale (Jacod and Shiryaev, 2013)
A process $\{X_t\}$ is said to be a **continuous semimartingale** if it can be decomposed into two adapted, *Cádlág* process M_t and A_t where X_t is given as:

$$X_t = M_t + A_t \tag{3.39}$$

where A_t is a **locally finite variation process** and M_t is a **local martingale** with $A(0) = M(0) = 0$

Definition 3.39: Stochastic volatility semimartingale (Jacod and Shiryaev, 2013) Stochastic volatility semimartingale (Svsm) which is referred to as Brownian semimartingale by Barndorff-Nielsen and Shephard (2006) is a semimartingale X_t which can be decomposed into two adapted and *Cádlág* processes: $X_t = M_t + A_t$ where, M_t is a continuous *Itô* stochastic integral of spot volatility process, σ_t with respect to the standard Brownian motion W_t, that is: $M_t = \int_0^t \sigma_s dW_s$ Given that the spot volatility process $\sigma_t > 0$ is with paths of finite variation, adapted, *Cádlág* and bounded away from zero. Also, A_t is continuous, and is the Riemann integral of α_t (drift process), where α_t is an adapted process with paths of finite variation, that is, $A_t = \int_0^t \alpha_s ds$. The above special kind of semimartingale is said to belong to a class of continuous stochastic volatility semimartingale (denoted as $Svsm^c$ in this thesis). Thus, if $X_t \in Svsm^c$, then X_t can be expressed as:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s \tag{3.40}$$

where, α_t *and* σ_t are respectively the drift and the volatility process and W_t is standard Brownian motion process.

Definition 3.40: Purely discontinuous semimartingale (He *et al.*, 2018)
A purely discontinuous semimartingale process X_t is a process whose quadratic variation denoted by $[X]$ are pure jump processes.
That is,

$$[X]_t = \sum_{s \leq t} \Delta X_s^2$$

where, ΔX_s is the size of jump of the process at time s.

Definition 3.41: Continuous stochastic volatility semimartingale with jumps ($Svsm^j$)
Given that X_t is the log return of the stock price with a continuous part X_t^c and a part X_t^j, That is, $X_t = X_t^c + X_t^j$, where, $X_t^c \in Svsm^c$ expressed as: $X_t^c = \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s$ and X_t^j is the discontinuous (jump) part defined as: $X_t^j = \sum_{i=1}^{N(t)} Q_j$ where $N(t)$ is a simple counting process which stands for the number of jumps at time t and Q_j is a non- zero stochastic process. Then, X_t is said to belong to a class of stochastic volatility semimartingale with added jumps

$(X_t \in Svsm^j)$. Thus,

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N(t)} Q_j \tag{3.41}$$

with the processes: $\alpha_t, \sigma_t, W_t, Q_j \ and \ N(t)$ as defined above.

3.4 STUDY THREE

The realised power variation and the realised multipower variation process

In this study, basic concepts of the realised power variation, the quadratic variance and the realised multipower variation process are discussed. More so, the asymptotic properties of the difference of the quadratic variation, the realised multipower variation and the realised variance were presented.

3.4.1 Realised power variation, quadratic variance and realised variance

Given a positive real constant ($r > 0$), the description in equation (1.6) on the concepts of the r^{th}-realised power variation, realised variance, quadratic variation and the multipower variation process is employed.

Definition 3.44: The r^{th}-order realised power variation (Barndorff-Nielsen and Shephard, 2003b)

The r^{th}-order RPV of the process X_t is defined as:

$$\{X_\Delta\}_t^{(r)} = \Delta^{(1-\frac{r}{2})} \sum_{j=1}^{[\frac{t}{\Delta}]} |x_j|^r \tag{3.42}$$

where, r is a positive real constant $\Delta^{(1-\frac{r}{2})}$ is the normalization in power variation, the increasing sequence of discrete times $0 = t_0 < t_1 < \cdots < t_n = t$, $[\frac{t}{\Delta}]$ is an integer $\implies [\frac{t}{\Delta}] = n$ for simplicity, and the log return of stock price at the j^{th}-time is given as:

$$\mid x_j \mid = \mid X_{t_j} - X_{t_{j-1}} \mid = \mid X_{j\Delta} - X_{(j-1)\Delta} \mid \tag{3.43}$$

Definition 3.45: The r^{th}-Order Power Variation (Barndorff-Nielsen and Shephard, 2003b)

The r^{th}-order variation $\{X\}_t^{(r)}$ is defined as: the probability limit as $\Delta \to 0$ (implies $n \to \infty$) of the r^{th}-order realised power variation process.

That is,

$$\begin{aligned} \{X\}_t^{(r)} &= \mathbb{P} - \lim_{\Delta \to 0} \{X\}_{\Delta,t}^{(r)} \\ &= \mathbb{P} - \lim_{\Delta \to 0} \Delta^{(1-\frac{r}{2})} \sum_{j=1}^{[\frac{t}{\Delta}]} |X_{j\Delta} - X_{(j-1)\Delta}|^r \end{aligned} \tag{3.44}$$

where, $r, \Delta^{(1-\frac{r}{2})}, |x_j|^r, [\frac{t}{\Delta}]$ *and* $X_{j\Delta}$ are as defined above.

Remark 3.10 (Barndorff-Nielsen and Shephard, 2003b)

In the sequel, the convergence results of the above defined process in a class of continuous stochastic volatility semimartingales ($Svsm^c$) was considered

Theorem 3.7 (Barndorff-Nielsen and Shephard, 2003b)

Let X_t be the log returns of the stock price where $X_t \in (Svsm^c)$ as defined in Definition 3.38. Let's assume that the drift parameter α_t is zero ($\alpha_t = 0$) and the volatility σ_t is independent of the Brownian Motion W_t. Then,

$$\{X\}_t^{(r)} = \mu_r \int_0^t \sigma_s^r ds, \tag{3.45}$$

where, $\mu_r = \mathbb{E}(|\nu|^r) = \frac{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+\frac{1}{2}\right)}{\sqrt{\pi}}, r > 0, \nu \sim N(0,1), \quad \nu_i$ are independent and identically distributed.

Definition 3.46: Quadratic Variation (Barndorff-Nielsen and Shephard, 2002a)

The Quadratic Variation (QV) process of the log returns X_t of the stock price is defined as:

$$\begin{aligned} [X]_t &= \mathbb{P} - \lim_{n\to\infty} \left(\sum_{j=1}^{n} \left(X_{t_j} - X_{t_{j-1}} \right)^2 \right) \\ &= \mathbb{P} - \lim_{n\to\infty} \sum_{j=1}^{n} |x_j|^2 \end{aligned} \tag{3.46}$$

where, $x_j = X_{t_j} - X_{t_{j-1}} = X_{j_\Delta} - X_{(j-1)\Delta}$ since $t_j = j\Delta$ for $0 = t_0 < t_1 < \ldots < t_n = t$ with $max_j \Big(j\Delta - (j-1)\Delta \Big) \to 0$ as $n \to \infty$.

Also, the realised QV process $[X]_{\Delta,t}$ is defined as:

$$\begin{aligned} [X]_{\Delta,t} &= \sum_{j=1}^{n} (x_j)^2 \\ &= \sum_{j=1}^{n} (X_{t_j} - X_{t_{j-1}})^2 \end{aligned} \tag{3.47}$$

Definition 3.47: Realised Variance (RV) Process (Barndorff-Nielsen and Shephard, 2002a, Barndorff-Nielsen and Shephard, 2003b, Barndorff-Nielsen and Shephard, 2004). The realised variance (RV) process of the log returns X_t of stock price (a semimartingale process) is defined as:

$$[X]^{(2)}_{\Delta,t} = \sum_{j=1}^{\frac{t}{\Delta}} x_j^2 \tag{3.48}$$

Remark 3.11

The following theorem shows the relationship between the QV and RV processes.

Theorem 3.8 (Barndorff-Nielsen and Shephard, 2003a)

Let $X_t \in Svsm^c$ with its dynamics given as:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s \tag{3.49}$$

where, $A_t = \int_0^t \alpha_s ds$ is the drift process (a finite variation process) and $M_t = \int_0^t \sigma_s dW_s$ is the stochastic volatility process (a *Cádlág* adapted process). Then the quadratic variation (QV) process of X_t converges in probability to the realised variance (RV) of X_t which is an estimator of the integrated volatility process.

Hence,

$$[X]_t \xrightarrow{\mathbb{P}} [X]^{(2)}_{\Delta,t} = \sum_{j=1}^{\frac{t}{\Delta}} x_j^2 \xrightarrow{\mathbb{P}} \int_0^t \sigma_s ds \tag{3.50}$$

Remark 3.12

The asymptotic property of the difference between the RV process and the QV process is given in the next theorem without proof. See also, Barndorff-Nielsen *et al.* (2006c) and Barndorff-Nielsen *et al.* (2006a).

Theorem 3.9 (Barndorff-Nielsen and Shephard, 2003b)

Let $X_t \in Svsm^c$, such that, $X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s$, given that the processes $\alpha_t,\ and\ \sigma_t$ satisfies the following weak conditions (Barndorff-Nielsen and Shephard, 2003b):

1. The volatility process σ_t is (pathwise) locally bounded away from 0 and satisfies the property

$$\lim_{\Delta \to 0} \left(\Delta^{\frac{1}{2}} \sum_{j=1}^{[\frac{t}{\Delta}]} \left| \omega^2_{\xi_j} - \omega^2_{\theta_j} \right| \right) = 0 \tag{3.51}$$

given that $\omega_t^2 = \int_0^t \sigma_s^2 ds$ for any $\xi_j = \xi_j(\Delta)$ and $\theta_j = \theta_j(\Delta)$ such that

$$0 \leq \xi_1 \leq \theta_1 \leq \Delta \leq \xi_2 \leq \theta_2 \leq 2\Delta \leq \ldots \leq \xi_n \leq \theta_n \leq n\Delta = t$$

2. The drift(mean) process α_t is continuous, satisfying:

$$\lim_{\Delta \to 0} \left(\max_{1 \le j \le n} \Delta^{-1} \left| \alpha_{j\Delta} - \alpha_{(j-1)\Delta} \right| \right) < \infty \tag{3.52}$$

Then,

$$\frac{\Delta^{\frac{1}{2}} \left([X]^{(2)}_{\Delta,t} - [X]_t \right)}{\sqrt{\omega_t^4}} \xrightarrow{L} \widetilde{\mathrm{V}} \tag{3.53}$$

where $\widetilde{\mathrm{V}} \sim N(0, \varphi_{RV})$, $\omega_t = \int_0^t \sigma_s ds$ and $\varphi_{RV} = Var(|\nu|^2), \nu \sim N(0,1)$. Thus by calculation $\varphi_{RV} = 2$

Theorem 3.10: QV of $Svsm^c$ plus jumps (Barndorff-Nielsen *et al.*, 2006a)
Let $X_t \in Svsm^j$ such that X_t is given as:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N(t)} Q_j \tag{3.54}$$

where α_t is a finite variation process, *Cádlág* and adapted, σ_t is locally bounded away from zero, *Cádlág* and adapted, N_t is a simple counting process for all t and Q_j is a positive random variable.
Then,

$$[X]_t = \int_0^t \sigma_s^2 ds + \sum_{j=1}^{N_t} Q_j^2 \tag{3.55}$$

Remark 3.13
The result given in theorem (3.10) indicates that when jumps are added to a class of continuous stochastic volatility semimartingale process, the quadratic variation gives the sum of the integrated volatility and the sum of the squares of the jumps in the process.

3.4.2 Realised multipower variation process

In financial modelling, to determine the right dynamics for a process, a major feature to consider in any stock process under observation is the “presence” or “absence” of jumps in the process. In the previous subsection, it was observed that there is a relationship between the RV and the QV processes in the class of $Svsm^c$ process. The question as to what happens to this process when jumps are added to this class was considered in this subsection.

In view of the above, a more robust process- the Realised Multipower Variation (RMPV) and its asymptotic properties were presented in this subsection. Considering the discrete observation described in subsection 3.4.1, the (MPV) process and its realised version are defined below.

Definition 3.48: Realised multipower variation (RMPV) process

The realised multipower variation process defined on a one-dimensional semimartingale process in its generalised form is given as:

$$\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)} = \Delta^{1-\delta(r_1,\ldots,r_m)} \sum_{j=1}^{c(t,m,\Delta)} f(x_j, r_i). \tag{3.56}$$

as defined in Barndorff-Nielsen *et al.* (2006d),,

where $\delta(r_1,\ldots,r_m) = \frac{1}{2}\sum_{i=1}^{m} r_i$, $c(t,m,\Delta) = [{}^{t}/_{\Delta}] - (m-1)$ and $f(x_j,r_i) = \prod_{i=0}^{m-1} \left|x_{j+i}\right|^{r_{i+1}}$ for $n > m$. The asymptotic properties of equation (3.56) above, were extensively given in Barndorff-Nielsen *et al.* (2006b) and Kinnebrock and Podolskij (2008). Also, the multipower variation of the process X_t is defined as: the probability limit (as $\Delta \to 0 \implies n \to \infty$) of the realised MPV process.

$$\begin{aligned}\{X\}_{t}^{[r_1,r_2,\ldots,r_m]} &= \mathbb{P} - \lim_{\Delta\to 0}\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_m)} \\ &= \mathbb{P} - \lim_{\Delta\to 0}\left(\Delta^{1-\delta(r_1,\ldots,r_m)} \sum_{j=1}^{c(t,m,\Delta)} f(x_j, r_i)\right)\end{aligned} \tag{3.57}$$

with $\delta(r_1, r_2, \ldots, r_m)$, n and m as defined above.

Remark 3.14

1. The first (1^{st}) order particular case of the above defined realised multipower variation process is the Bipower variation (BVP) process defined for $m = 2$ as:

$$\{X\}_{\Delta,t}^{[r_1,r_2]} = \Delta^{1-\delta(r_1,r_2)} \sum_{j=2}^{n} \left(|x_j|^{r_1}|x_{j-1}|^{r_2}\right) \tag{3.58}$$

with $\delta(r_1, r_2) = \frac{1}{2}(r_1 + r_2)$, $n = [\frac{t}{\Delta}]$ an integer and $n > 2$.

2. There are other particular cases of the multipower variation process that were studied in literature: the tripower (TP) variation process, and the quadpower variation process (QPV) as can be found in Barndorff-Nielsen and Shephard (2006) and Barndorff-Nielsen *et al.* (2006a). were shall extend in this thesis the existing results to particular cases of higher order: $m = 5, 6, 7, 8, 9, 10$ as Pentpower variation (PPV), Hexpower variation (HPV), Heptpower variation (HV), Octpower variation

(OPV), Nonpower variation (NPV), and Decpower variation (DPV) respectively in the next chapter.

3. In the next theorem, the convergence result of a special case of the Bipower variation (BPV) process, (that is, where $r_1 = r_2 = 1$), which proves robust to jumps was presented. Given that:

$$\{X\}_t^{[1,1]} = \mathrm{P} - \lim_{\Delta \to 0} \sum_{j=2}^{n} |x_j||x_{j-1}| \tag{3.59}$$

Theorem 3.11 (Barndorff-Nielsen and Shephard, 2006)

Let $X_t \in Svsm^j$ such that X_t is defined as:

$$X_t = \int_0^t \alpha_s ds + \int_s \sigma_s dW_s + \sum_{j=1}^{N_t} Q_j \tag{3.60}$$

That is, $X_t = X_t^c + X_t^j$, where, $X_t^c = \int_0^t \alpha_s ds + \int_s \sigma_s dW_s$ is the continuous part of X_t and, $X_t^j = \sum_{j=1}^{N_t} Q_j$ is the jump part of X_t. Then the $(1,1)$-Bipower variation process is:

$$\{X\}_t^{(1,1)} = \mu_1^2 \int_0^t \sigma_s^s ds \tag{3.61}$$

where $\mu_1 = \mathbb{E}(|\nu|) = 2^{\frac{1}{2}}(\pi)^{-0.5}$, where, $\nu \sim N(0,1)$.

Remark 3.15

If $X_t \in Svsm^j$, then the QV of the process X_t is the sum of the integrated volatility $\int_0^t \sigma_s ds$ and the sum of the squares of the jumps in X_t as shown in theorem (3.7). In summary,

$$[X]_t = \int_0^t \sigma_s ds + \sum_{j=1}^{N_t} Q_j^2$$

and

$$\mu_1^{-2}\{X\}_t^{(1,1)} = \int_0^t \sigma_s ds$$

By subtracting the above two expressions, gives:

$$[X]_t - \mu_1^{-2}\{X\}_t^{(1,1)} = \sum_{j=1}^{N_t} Q_j^2 \tag{3.62}$$

Thus, equation (3.62) forms the basis for the BNS jump test method proposed by Barndorff-Nielsen and Shephard (2006). Details of this method will be discussed in the next section of this chapter.

3.4.3 Asymptotic properties of the realised multipower variation process

Having defined the realised multipower variation process in definition (3.48), the asymptotic properties of the process were presented here. These comprise the convergence in probability result of the multipower variation process as can be found in Barndorff-Nielsen and Shephard (2006) and Barndorff-Nielsen *et al.* (2006c), as well as the convergence in law (distribution) result of the difference of the realised variance and the realised multipower variation process in the class of $Svsm$ as can be found in Barndorff-Nielsen and Shephard (2006) and Ysusi (2010). These theories form the basis for the jump test analysis in the next chapter.

3.4.4 Convergence in probability of the realised multipower variation process

In subsection 3.4.1, the realised r^{th} order variation process $\{X\}_{\Delta,t}^{(r)}$ was shown to converge in probability to the integrated volatility process. Here, the same result is given for a generalised multipower variation case via same method used in subsection 3.4.1.

Theorem 3.12 (Barndorff-Nielsen *et al.*, 2006a, Ysusi, 2006)
Let the process X_t belong to a class of continuous stochastic volatility semimartingale ($Svsm^c$) processes, such that X_T is defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and can be expressed as:

$$X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s \tag{3.63}$$

where, W_t is a Brownian motion process, α_t is a locally bounded and predictable drift process and σ_t is an adapted, $C\acute{a}dl\acute{a}g$ volatility process. Then, for $\alpha_t = 0$, the multipower variation process:

$$\{X\}_t^{[r_1,\ldots,r_m]} \xrightarrow{\mathbb{P}} \mu_{r_1}, \ldots, \mu_{r_i} \int_0^t |\sigma_s|^{2(\delta(r_1,\ldots,r_m))} ds$$

,

where,

$$2\delta(r_1, \ldots, r_m) = \sum_{i=1}^{m} r_i, \quad \mu_{r_i} = \mathbb{E}(|\nu|^r), \ \nu \sim N(0,1)$$

Proof:

Given that $X_t \in Svsm^c$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, W_t is a Brownian motion process, α_t is a locally bounded and predictable drift process and σ_t is an adapted, *Cádlág* volatility process. Therefore, for, $\alpha_t = 0$, then:

$$\{X\}_t = \{M\}_t \tag{3.64}$$

Thus, the expression in equation (3.63) based on the Markov inequality condition, goes to zero as $n \to \infty$ $(\Delta \to 0)$. That is, the expectation of

$$\left| \{M\}_{\Delta,t}^{[r_1,\ldots,r_m)} - (\mu_{r_1}, \ldots, \mu_{r_m}) \int_0^t |\sigma_s|^{2(\delta(r_1,\ldots,r_m))} ds \right| \tag{3.65}$$

goes to zero. Given a simple volatility process σ_t defined by

$$\sigma_t = \sum_{j=1}^{k} \psi_j 1_{(t_{j-1},t_j]} \tag{3.66}$$

where, $0 = t_0 < t_1 < t_2 < \ldots < t_k = t$ and ψ_j is $\mathcal{F}_{t_{j-1}}$-measurable and bounded. Then, the stochastic integral of σ_t is given as:

$$\mathrm{I}(\sigma_t) = M_t = \sum_{j=1}^{k} \psi_j \big(W_{t_j} - W_{t_{j-1}} \big) \tag{3.67}$$

Considering the sub intervals $(t_0, t_1], (t_1, t_2], \ldots, (t_k, t_{k-1}]$ of $[0, t]$, the realised multipower variation process for $\{M\}_t$ in each of the subintervals is constructed below. Hence, from the above, the realised multipower variation process $\{M\}_{\Delta,t_1}^{[r_1,\ldots,r_m]}$, for the interval $(0, t_1]$ is defined as:

$$\{M\}_{\Delta,t_1}^{[r_1,\ldots,r_m)} = \Delta^{1-\delta(r_1,\ldots,r_m)} \sum_{j=m}^{[\frac{t_1}{\Delta}]} \left(\prod_{i=0}^{m-1} \left| \psi_1 \big(W_{t_{j-1}} - W_{t_{j-i-1}} \big) \right|^{r_i+1} \right) \tag{3.68}$$

where, $n' = [\frac{t_1}{\Delta}], n' > m'$ *and* $\delta(r_1, \ldots, r_m') = \frac{1}{2} \sum_{r=1}^{m'} r_i$

Then, the expectation of the above is obtained as:

$$\mathbb{E}\big(\{M\}_{\Delta,t_1}^{[r_1,\ldots,r'_m)}\big) = \mathbb{E}\Bigg(\Delta^{1-\delta(r_1,\ldots,r_m)}\sum_{j=m}^{[\frac{t_1}{\Delta}]}\Big(\prod_{i=0}^{m'-1}\big|\psi_1\big(W_{t_{j-1}} - W_{t_{j-i-1}}\big)\big|^{r_i+1}\Big)\Bigg)$$

$$= \Delta^{1-\delta(r_1,\ldots,r'_m)}\mathbb{E}\Bigg(\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\Big(\prod_{i=0}^{m'-1}\big|\psi_1\big(W_{t_{j-1}} - W_{t_{j-i-1}}\big)\big|^{r_i+1}\Big)\Bigg)$$

$$= \Delta^{1-\delta(r_1,\ldots,r'_m)}\Bigg(\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\mathbb{E}\Big(\prod_{i=0}^{m'-1}\big|\psi_1\big(W_{t_{j-1}} - W_{t_{j-i-1}}\big)\big|\Big)\Bigg)^{r_i+1}$$

since $|\psi_1(W_{t_{j-1}} - W_{t_{j-i-1}}|)^{r_{i+1}}$ for $i = 0, \ldots, m-1$ are independent increments and identically distributed standard Brownian motion, then,

$$\mathbb{E}\Bigg(\prod_{i=0}^{m'-1}\big|\psi_1\big(W_{t_{j-1}} - W_{t_{j-i-1}}\big)\big|^{r_i+1}\Bigg) = \mathbb{E}\Bigg(\prod_{i=0}^{m'-1}\big|\nu\sqrt{t_{j-1} - t_{j-i-1}}\big|^{r_{i+1}}\Bigg)$$

where, $\nu(\cdot) \sim N(0,1)$. Thus,

$$\mathbb{E}\big(\{M\}_{\Delta,t_1}^{[r_1,\ldots,r_{m'}]}\big) = \Delta^{1-\delta(r_1,\ldots,r_m)}\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\Bigg(\prod_{i=0}^{m'-1}|\psi_1|^{r_{i+1}}\mathbb{E}\Big(\prod_{i=0}^{m'-1}\big|\nu\sqrt{t_{j-1} - t_{j-i-1}}\big|^{r_{i+1}}\Big)\Bigg)$$

$$= \Delta^{1-\delta(r_1,\ldots,r_m)}\sum_{j=m}^{[\frac{t_1}{\Delta}]}\psi_1^{2\delta(r_1,\ldots,r_m)}\mathbb{E}\Big(\prod_{i=0}^{m'-1}\big|\nu\sqrt{t_{j-1} - t_{j-i-1}}\big|^{r_{i+1}}\Big)$$

$$= \Delta^{1-\delta(r_1,\ldots,r_m)}\sum_{j=m}^{[\frac{t_1}{\Delta}]}\psi_1^{2\delta(r_1,\ldots,r_m)}\mathbb{E}\Big(\prod_{i=0}^{m'-1}|\nu|^{r_{i+1}}\Big)\mathbb{E}\Big(\big|\sqrt{t_{j-1} - t_{j-i-1}}\big|^{r_{i+1}}\Big)$$

But, $t_{j-1} - t_{j-i-1} = \Delta$ which is positive and deterministic for $j = 1, \ldots, k$ and $i = 0, \ldots, m-1$ (since the subintervals are of equal distance).
Then,

$$\mathbb{E}\big(\{M\}_{\Delta,t_1}^{(r_1,\ldots,r_m)}\big) = \Delta^1\Delta^{-\delta(r_1,\ldots,r_m)}\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\psi_1^{2\delta(r_1,\ldots,r_m)}\Delta^{\delta(r_1,\ldots,r_m)}\prod_{i=0}^{m-1}\Big(\mathbb{E}|\nu|^{r_{i+1}}\Big)$$

$$= \prod_{i=0}^{m-1}\Big(\mathbb{E}|\nu|^{r_{i+1}}\Big)\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\psi_1^{2\delta(r_1,\ldots,r_m)}\Delta \tag{3.69}$$

$$= \prod_{i=0}^{m-1}\mu_{r_i}\sum_{j=m'}^{[\frac{t_1}{\Delta}]}\psi_1^{2\delta(r_1,\ldots,r_m)}\Delta$$

where, $\mu_{r_i} = \mathbb{E}|\nu|^{r_i}$ and (3.3.28) gives the expectation of the process $\{M\}_{\Delta,t_1}^{(r_1,\ldots,r_m)}$

for the subinterval $(0, t_1]$.

Hence, for the subinterval $(t_1, t_2]$,

$$\mathbb{E}\big(\{M\}_{\Delta,t_2}^{(r_1,\ldots,r_m)}\big) = \prod_{i=1}^{m} \mu_{r_i} \sum_{j=m}^{[\frac{t_1}{\Delta}]} \psi_1^{2\delta(r_1,\ldots,r_m)} \Delta \tag{3.70}$$

To obtain the realised multi power variation for the interval $[0, t]$, the sum of the above for all k subintervals is taken. Thus,

$$\mathbb{E}\big(\{M\}_{\Delta,t}^{(r_1,\ldots,r_m)}\big) = \prod_{i=1}^{m} \mu_{r_i} \sum_{i=1}^{k} \phi_i^{2\delta(r_1,\ldots,r_m)} \Delta \tag{3.71}$$

Taking the limit of both sides as $\Delta \to 0$ gives:

$$\begin{aligned}\lim_{\Delta\to 0} \mathbb{E}\big(\{M\}_{\Delta,t}^{(r_1,\ldots,r_m)}\big) &= \lim_{\Delta\to 0} \prod_{i=1}^{m} \mu_{r_i} \sum_{i=1}^{k} \phi_i^{2\delta(r_1,\ldots,r_m)} \Delta \\ &= \prod_{i=1}^{m} \mu_{r_i} \int_0^t \sigma_s^{2\delta(r_1,\ldots,r_m)} ds\end{aligned} \tag{3.72}$$

This implies that:

$$\lim_{\Delta\to 0} \left(\mathbb{E}\big(\{M\}_{\Delta,t}^{(r_1,\ldots,r_m)}\big) \right) - \prod_{i=1}^{m} \mu_{r_i} \int_0^t \sigma_s^{2\delta(r_1,\ldots,r_m)} ds = 0 \tag{3.73}$$

which also implies that the limit of the absolute value of the expression in equation (3.73) is also zero. This implies limit in probability. Therefore,

$$\begin{aligned}\{M\}_t^{(r_1,\ldots,r_m)} =& \mathbb{P} - \lim_{\Delta\to 0} \{M\}_t^{(r_1,\ldots,r_m)} \\ =& \mu_{r_1}, \ldots, \mu_{r_m} \int_0^t \sigma_s^{2\delta(r_1,\ldots,r_m)} ds\end{aligned} \tag{3.74}$$

Under the condition that $\alpha_t = 0$, Then,

$$\{X\}_t^{(r_1,\ldots,r_m)} = \{M\}_t^{(r_1,\ldots,r_m)} \tag{3.75}$$

Hence, from equations (3.74) and (3.75),

$$\{X\}_t^{(r_1,\ldots,r_m)} = \prod_{i=1}^{m} \mu_{r_i} \int_0^t \sigma_s^{2\delta(r_1,\ldots,r_m)} ds \tag{3.76}$$

where, $\delta(r_1,\ldots,r_m)=\frac{1}{2}\sum_{i=1}^{m} r_i; \mu_{r_i}=\mathbb{E}(|\nu|^{r_i})$ and $\nu \sim N(0,1)$

3.4.4.1 Asymptotic theory for the difference between the RMPV and the RV process

The linear jump test of the BNS method for detecting jumps in a high frequency financial data in Barndorff-Nielsen and Shephard (2006) is basically derived from the asymptotic distribution of a difference between the RV process and a particular case of the RMPV processes, specifically for $m = 2$ and for $r_1 = r_2 = 1$.

The generalised case of the above description was considered here. Hence, the need to study the asymptotic behaviour of the difference of the RV process and the RMPV processes with an intention of investigating the robustness of such processes.

In the next theorem, the asymptotic property of the difference between the RV process and the RMPV processes is stated; making reference to the working paper of Ysusi (2006), the work of Barndorff-Nielsen and Shephard (2006) and Barndorff-Nielsen *et al.* (2006a).

Theorem 3.13 (Barndorff-Nielsen *et al.* (2006c), Barndorff-Nielsen and Shephard (2002a) and Barndorff-Nielsen *et al.*, 2006a)

Let $X_t \in Svsm^c$, then as $\Delta \to 0$, and $A_t = 0$

$$\frac{1}{\sqrt{\Delta\int_0^t \sigma^4 ds}}\left(\mu_{\frac{2}{m}}^{-m}\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)} - \left[X\right]_{\Delta,t}^{(2)}\right) \xrightarrow{L} N(0,\varphi_{MPV}) \tag{3.77}$$

where, $\varphi_{MPV})$ is the asymptotic variance of the multipower variation process given as: $\varphi_{MPV}(m,v_i)=Var(|\nu|^2)+\mu_{\frac{2}{m}}^{-2m}\omega_m^2-\mu_{\frac{2}{m}}^{-m} mCov\Big(|\nu|^2,\prod_{i=1}^m|\nu_i|^{\frac{2}{m}}\Big)$, $\mu_{\frac{2}{m}}=$ $\mathrm{E}(|\nu|^{\frac{2}{m}})=\frac{2^{\frac{1}{m}}\Gamma(m+\frac{1}{2})}{\sqrt{\pi}}$ and $\omega_m^2=Var\big(\prod_{i=1}^m|\nu_i|^{\frac{2}{m}}\big)+2\sum_{j=1}^{m-1}Cov\Bigg(\prod_{i=1}^m|\nu_i|^{\frac{2}{m}},\prod_{i=1}^m|\nu_{i+j}|^{\frac{2}{m}}\Bigg)$

Remark 3.16

The asymptotic property of the difference between the RMPV and the RV processes will form the basis for the jump test of stock indices data in the next chapter of this thesis.

3.5 STUDY FOUR

Basic Methods

In this study, the BNS jump test for stock indices data set, as well as some stochastic methods for determining the solutions of the dynamics of a *Lévy*-jump diffusion models that were used in literature, were reviewed. The above-mentioned formed the methodology of this work.

3.5.1 The BNS jump test method in stock data

The BNS method for jump test was proposed by Barndorff-Nielsen and Shephard (2006) for $X_t \in Svsm^c$ based on the assumption that the volatility process σ_t^2 is pathwise bounded away from zero and independent of the Brownian process W_T. This method was basically derived from the asymptotic distribution of the difference between a particular realised bipower variation process: $\{X\}_{\Delta,t}^{(1,1)}$ and the realised variance process $[X]_{\Delta,t}^{(2)}$.
That is, for $m = 2$, and $r_1 = r_2 = 1$ then,

$$\frac{\Delta^{\frac{-1}{2}}\left(\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)} - [X]_{\Delta,t}^{(2)}\right)}{\sqrt{\int_0^t \sigma_s^4 ds}} \xrightarrow{L} N\Big(0, \varphi_{BPV}\Big) \tag{3.78}$$

where, $\varphi_{BPV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.6091, \mu_1 = \frac{\sqrt{2}}{\sqrt{\pi}}$.

The BNS jump-test is classified into the feasible linear jump test, the ratio-jump test and the adjusted ratio test. The linear test is based on the result given in equation (3.78), while the ratio test is obtained by dividing the numerator and the denominator of equation (3.78) by $[X]_{\Delta,t}^{(2)}$ to obtain an infeasible ratio test given as:

$$\frac{\Delta^{\frac{-1}{2}}\left(\frac{\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)}}{\mu_1^{-2}[X]_{\Delta,t}^{(2)}} - 1\right)}{\sqrt{\frac{\int_0^t \sigma_s^4 d}{(\int_0^t \sigma_s^4 ds)^2}}} \xrightarrow{L} N\Big(0, \varphi_{BPV}\Big) \tag{3.79}$$

In a real life high frequency financial data, the quantity $\int_0^t \sigma_s^4 ds$ is inestimable and as such, an estimator for $\int_0^t \sigma_s^4 ds$ is needed to obtain a feasible test. Thus by replacing the quantity $\int_0^t \sigma_s^4 ds$ by the realised quadpower variation $\mu_1^{-4}\{X\}_{\Delta,t}^{[1,1,1,1]}$

and $\int_0^t \sigma_s^2 ds$ is replaced by $\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)}$. Since,

$$\{X\}_{\Delta,t}^{[1,1,1,1]} = \sum_{j=4}^{[\frac{t}{\Delta}]} |x_j||x_{j-1}||x_{j-2}||x_{j-3}| \xrightarrow{\mathbb{P}} \mu_1^4 \int_0^t \sigma_s^2 ds \tag{3.80}$$

and

$$\{X\}_{\Delta,t}^{(1,1)} = \sum_{j=4}^{[\frac{t}{\Delta}]} |x_j||x_{j-1}| \xrightarrow{\mathbb{P}} \mu_1^2 \int_0^t \sigma_s^2 ds \tag{3.81}$$

Then, the feasible linear jump test, ratio test and adjusted ratio test were given respectively as:

$$\hat{P} = \frac{\Delta^{\frac{-1}{2}}\Big(\mu_1^2\{X\}_{\Delta,t}^{(1,1)} - [X]_{\Delta,t}^{(2)}\Big)}{\sqrt{\mu_1^4\{X\}_{\Delta,t}^{[1,1,1,1]}}} \xrightarrow{L} N\Big(0, \varphi_{BPV}\Big) \tag{3.82}$$

$$\hat{Q} = \frac{\Delta^{\frac{-1}{2}}\Big(\mu_1^2\{X\}_{\Delta,t}^{(1,1)}/[X]_{\Delta,t}^{(2)} - 1\Big)}{\sqrt{\{X\}_{\Delta,t}^{[1,1,1,1]}/\Big(\{X\}_{\Delta,t}^{(1,1)}\Big)^2}} \xrightarrow{L} N\Big(0, \varphi_{BPV}\Big) \tag{3.83}$$

$$\hat{R} = \frac{\Delta^{\frac{-1}{2}}\Big(\mu_1^2\{X\}_{\Delta,t}^{(1,1)}/[X]_{\Delta,t}^{(2)} - 1\Big)}{\max\Big(\frac{1}{t}, \sqrt{\{X\}_{\Delta,t}^{[1,1,1,1]}/\Big(\{X\}_{\Delta,t}^{(1,1)}\Big)^2}\Big)} \xrightarrow{L} N\Big(0, \varphi_{BPV}\Big) \tag{3.84}$$

where, $\varphi_{BPV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \simeq 0.6091, \mu_1^2 = \frac{\pi}{2}$.

The hypotheses for the test are:

$$H_0 : X_t \in Svsm^c \tag{3.85}$$

$$H_1 : X_t \in Svsm^j \tag{3.86}$$

3.5.2 Stochastic formula for solving the dynamics of jump-diffusion models

Here, stochastic formula for solving the dynamics of diffusion and were processes, respectively, the $It\hat{o}'s$ formula for diffusion Process and jump-diffusion process were stated. These were given according to: Bichteler and Klaus (2002), Cont and Tankov (2004) and Oskendal and Sulem (2005); the $It\hat{o}'s$ formulae for diffusion and jump diffusion processes were stated without proof.

Theorem 3.14: *Itô's* formula for diffusion process (Bichteler and Klaus, 2002)
Let the SDE of the process $\{X_t\}_{t\geq 0}$ be given as:

$$dX_t = \alpha(t,\omega)X_t dt + \sigma(t,\omega)X_t dW_t \tag{3.87}$$

for, $t \geq 0$, with initial value $X_0 = 0$ where $\alpha(t,\omega)$ *and* $\sigma(t,\omega)$ are two predictable stochastic processes.
Consider a function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, which is differentiable with respect to t and twice continuously differentiable with respect to x which implies the functions $\partial f(t), \partial f(x)$ *and* $\partial^2 f(x)$ exist and are continuous. Then, the stochastic differential of the process: $Y(t) = f(t, X(t))$ is given as:

$$\begin{aligned} dY(t) := df(t,X_t) = &\Big(\frac{\partial f(t,X_t)}{\partial t} + \alpha(t,\omega)X_t\frac{\partial f(t,X_t)}{\partial x} \\ &+ \frac{1}{2}\sigma^2(t,\omega)X_t^2\frac{\partial^2 f(t,X_t)}{\partial^2 x}\Bigg)dt + \sigma(t,\omega)X_t\frac{\partial f(t,X_t)}{\partial x}dW_t, \quad t \geq 0 \end{aligned}$$

Theorem 3.15: Itô's formula for ***Lévy*-** **jump processes** (Cont and Tankov, 2004)
Let X_t be a *Lévy* process with jumps given in its stochastic integral form:

$$X_t = X_0 + \int_0^t \alpha(s,\omega)ds + \int_0^t \sigma(s,\omega)dW_s + \int_0^t \int_{\mathbb{R}} \nu(s,\omega)N(ds,dx) \tag{3.88}$$

where,

$$\widetilde{N}(ds,dz) = \begin{cases} N(ds,dz) - \nu(dz)ds & \text{if} \quad |Z| < \mathbb{R} \\ \widetilde{N}(ds,dz) & \text{if} \quad |Z| \geq \mathbb{R} \end{cases} \tag{3.89}$$

for $\mathbb{R} \in [0,\infty)$, where $\widetilde{N}(ds,dz) - \nu(dz)ds$ is the compensated Poisson random measure of X_t *and* W_t is an independent Brownian motion.
Given the SDE of X_t as:

$$dX_t = \alpha(t,\omega)X_t dt + \sigma(t,\omega)X_t dW_t + \int_{\mathbb{R}} Q(t,z,\omega)N(dt,dz) \tag{3.90}$$

Let the function $Y_t = f(t, X_t)$ be differentiable with respect to t and twice continuously differentiable with respect to x, that is ($f \in C^2(\mathbb{R}^2)$).

Then,

$$
\begin{aligned}
dY(t) := df(t, X_t) = \Big(&\frac{\partial f(t, X_t)}{\partial t} + \alpha(t, \omega)\frac{\partial f(t, X_t)}{\partial x} \\
&+ \frac{1}{2}\sigma^2(t, \omega)\frac{\partial^2 f(t, X_t)}{\partial^2 x}\Bigg) dt + \sigma(t, \omega)\frac{\partial f(t, X_t)}{\partial x} dW_t \\
&+ \big(f(t, X_{t-} + \Delta X_t) - f(t, X_{t-})\big) dN_t, \quad t \geq 0
\end{aligned}
$$

3.6 STUDY FIVE

Basic Models: Stock price models

In this study, two categories of stock price models (dynamics of stock price) as well as their respective probability density functions were discussed. The first was in the category of the diffusion models-the Geometric Brownian Motion (GBM) model as in Black and Scholes (1973a); and the second category was the jump-diffusion models of different forms depending on the distributions of their jump amplitude which can be found in Hanson and Westman (2002), Duffie *et al.* (2000) and Kou (2002).

3.6.1 Geometric Brownian motion model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space as given in definition (3.9) and $\mathbb{F} = \{\mathcal{F}_t : t \in [0, \infty)\}$ a filtration satisfying the conditions in definition (3.10). Let $\tilde{S}_t$ be the price of stock at time t, which evolves according to the SDE:

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \tag{3.91}$$

The above dynamics is termed as the GBM model for the stock price $\tilde{S}_t$ at time t; where $S_0 \geq 0$, μ is the drift coefficient also known as as the expected rate of return of S and σ is the diffusion coefficient (volatility), W_t is a standard Brownian motion with respect to $\mathbb{F}$. The solution of the dynamics given in (3.91) via the *Itô*'s formula in Theorem 3.14 can be obtained. Equation (3.91) can be expressed as:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu dt + \sigma dW_t \tag{3.92}$$

$$\implies d(\ln S) = \mu dt + \sigma dW_t \tag{3.93}$$

Let $u(t, \tilde{S}_t) = \ln S$ then, $\frac{\partial u}{\partial t} = 0$, $\frac{\partial u}{\partial S} = \frac{1}{S}$ and $\frac{\partial^2 u}{\partial S^2} = \frac{-1}{S^2}$
Thus, by the *Itô*'s formula (Theorem 3.14),

$$u(t, \tilde{S}_t) = u(t_0, S_{t_0}) + \int_{t_0}^{t} \left(\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial S} + \frac{1}{2} f^2 \frac{\partial^2 u}{\partial S^2} \right) dt + \int_{t_0}^{t} f \frac{\partial u}{\partial S} dW_t \tag{3.94}$$

$$\tilde{S}_t = S_0 exp\Big((\mu - \frac{\sigma^2}{2})t + \sigma W_t \Big) \tag{3.95}$$

Suppose, $u \sim N(\mu, \sigma^2)$, then the probability density function of u is given as:

$$f(u) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}; u \in (-\infty, \infty), \mu \in (-\infty, \infty) \quad \sigma > 0 \tag{3.96}$$

The lognormal pdf of the stock price $\tilde{S}_t$ can be obtained as follows:

$$f(u)du = g(\tilde{S}_t)d\tilde{S}_t \implies \qquad g(\tilde{S}_t) = \frac{f(u)du}{d\tilde{S}_t} \tag{3.97}$$

Let, $u = ln\tilde{S}_t$ and $du = \frac{d\tilde{S}_t}{\tilde{S}_t}$ in equation ((3.96)),

$$g(\tilde{S}_t) = \frac{1}{\tilde{S}_t \sigma\sqrt{2\pi}} \mathrm{e}^{\frac{-1}{2}\left(\frac{\ln \tilde{S}_t - \mu}{\sigma}\right)^2} \tag{3.98}$$

3.6.2 Jump - diffusion models

Models derived from the diffusion process like the GBM model upon which the famous B-S model was built, have some deficiencies.

The distribution of the GBM model is a normal one; and this contradicts the empirical log returns of most financial data obtained from the stock market; it is always found to be negatively skewed (asymmetric) and with sharp peak (leptokurtic). In the B-S model, the implied volatility is found to be a constant instead of a convex function of a strike price that looks like a "Smile" (Synowiec, 2008). Also the GBM model is devoid of jumps, but most financial data in reality present jumps in their price processes.

In order to take care of the above-mentioned deficiencies of the GBM model, many efficient models were suggested in the past few years. These models include the jump - diffusion models like the AJD model in Duffie *et al.* (2000); which was able to capture the leptokurtic property of the process. A special form of the AJD model is the normal JD model given in Hanson and Westman (2002). Others include the Merton model in Merton (1976); Double exponential jump diffusion (DEJD) model in Kou (2002) and Uniform Jump Diffusion (UJD) Model (Hanson *et al.*, 2004).

A review of two of the above-mentioned, considering the densities of the log return of the stock-price process subject to the distributions of their jump amplitudes were considered.

3.6.3 The normal jump-diffusion model

The dynamics of the stock price $\tilde{S}_t$ (when jumps are present), defined on a filtered probability space $(\Omega, F, \mathbb{P}, \mathbb{F})$, were given as:

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \sigma_d \tilde{S}_t dW_t + \tilde{S}_t J(Q) dN_t \tag{3.99}$$

where, $S_0 > 0$, μ is mean return rate of the diffusive process, σ_d is the diffusive volatility, W_t is a standard Brownian Motion, N_t is a Poisson process with respect to the filtration $\mathbb{F}$ having a constant jump rate η, and $J(Q_j)$ is a non-constant jump amplitude. Note that in equation (3.99), the random variables, W_t, N_t and J(Q) are independent, and the jump process in the model is given as:

$$\int_{t_1}^{t_2} J(Q_j) dN_t = \sum_{i=1}^{N_{t_2-t_1}} J(Q_j), \quad t_2 > t_1 \tag{3.100}$$

such that the $Q_j's$ are $i.i.d$ random variables, the pdf of N_t is

$$\mathbb{P}(N_t = k) = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!} \tag{3.101}$$

The solution of the SDE with jumps in equation (3.99) via the It$\hat{o}$'s formula for jump-diffusion given in Theorem (3.15) can be determined.

Thus, equation (3.99) can be written as:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu dt + \sigma_d dW_t + J(Q) dN_t$$

$$\implies \quad d(\ln \tilde{S}_t) = \mu dt + \sigma_d dW_t + J(Q) dN_t \tag{3.102}$$

Let, $u(t, \tilde{S}_t) = \ln S$, $\implies \dfrac{\partial u}{\partial t} = 0, \dfrac{\partial u}{\partial x} = \dfrac{1}{S}$ and $\dfrac{\partial^2 u}{\partial S^2} = \dfrac{-1}{S^2}$

Thus,

$$d(\ln \tilde{S}_t) = (0 + \mu \tilde{S}_t \frac{1}{S} - \sigma^2 \frac{S_t^2}{2} \cdot \frac{-1}{S_t^2})dt + \sigma S \cdot \frac{1}{S} dW_t + \ln Q_t$$

$$d(\ln \tilde{S}_t) = (\mu - \frac{\sigma^2}{2})dt + \sigma dW_t + \ln Q_t$$

Integrating over $(t, t + \Delta t)$ gives

$$\ln S_{t+\Delta t} = \ln \tilde{S}_t + (\mu - \frac{\sigma_d^2}{2})t + \sigma_d W_t + \sum_{j=0}^{N_t} \ln Q_j$$

$$\implies \quad \ln(\frac{S_{t+\Delta t}}{\tilde{S}_t}) = (\mu - \frac{\sigma_d^2}{2})t + \sigma_d W_t + \sum_{j=0}^{N_t} \ln Q_j$$

For the interval (0,t),

$$\tilde{S}_t = S_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma_\mathrm{d}^2)\mathrm{t} + \sigma_\mathrm{d} \mathrm{W_t} + \sum_{\mathrm{j=0}}^{\mathrm{N_t}} \ln \mathrm{Q_j}}$$

Moreso, upon integration of equation (3.102), (3.103) was obtained as follows:

$$\Delta(\ln \tilde{S}_t) = (\mu - \frac{1}{2}\sigma_d^2)\Delta t + \sigma_d \Delta W_t + Q\Delta N_t \qquad (3.103)$$

where,

$$\Delta(\ln \tilde{S}_t) = \ln S_{t+\Delta t} - \ln \tilde{S}_t$$

$$\Delta W_t = W_{t+\Delta t} - W_t$$

$$\Delta N_t = N_{t+\Delta t} - N_t$$

Δt is a small increment change in time. It is clear from the above that the densities of the price process in equation (3.99) and the log returns process in equation (3.103) are basically determined by the distribution of the jump amplitude Q. In the next theorem, the probability density function of the process in equation ((3.103)), when Q is normally distributed as in Hanson and Westman (2002), was stated.

Theorem 3.16 (Hanson and Westman, 2002, Synowiec, 2008)
The **probability density** for the jump-diffusion log return $\Delta(\ln \tilde{S}_t)$ is given by

$$g_{\Delta(\ln \tilde{S}_t)}(x) = \sum_{k=0}^{\infty} \mathbb{P}_k(\lambda \Delta t)\phi(x_{i;} \;\; (\mu - \frac{1}{2}\sigma_d^2))\Delta t + \mu_j k, \;\; \sigma^2 \Delta t + \sigma_j^2 k) \qquad (3.104)$$

where $x \in \mathbb{R}$; $\mathbb{P}_k(\lambda \Delta t) = \mathbb{P}_k(N_t = k) = \frac{(\lambda t)^k \mathrm{e}^{-\lambda t}}{k!}$, and the normal density ϕ is given as:

$$g_{Q_j}(x) = \phi(x;\ \ \mu_j, \sigma_j^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{\frac{-(\mathrm{x}-\mu_{\mathrm{j}})^2}{2\sigma_{\mathrm{j}}}} \tag{3.105}$$

3.6.4 The double exponential jump-diffusion model (The Kou model)

In the next theorem, a particular jump diffusion driven by a double-exponential distribution was considered; this is called the **Kou Model.** The difference between the normal - jump diffusion process (model) and the Kou model is that the **upward** trend of jumps and the **downward** trend of jumps are treated separately in the Kou model such that the mean intensity of the upward jump and the downward jumps are respectively $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$.

Theorem 3.17 (Kou, 2002, Synowiec, 2008)
The probability density for the double exponential jump-diffusion log return $\Delta(\ln \tilde{S}_t)$ is given by

$$f_{\Delta(\ln \tilde{S}_t)}(x) = \frac{1-\lambda\Delta t}{\sigma\sqrt{\Delta t}} \phi\left(\frac{x - \left(\mu -^1/_2\sigma_d^2\right)\Delta t}{\sigma\sqrt{\Delta t}}\right) + \lambda\Delta t \left(M(x; \mu, \sigma_d, \eta_1, \eta_2, p, q)\right) \tag{3.106}$$

where,

$$\begin{aligned} M(x) =& p\eta_1 \mathrm{e}^{1/2\eta_1^2\sigma^2\Delta t} \mathrm{e}^{-\left(x-\left(\mu-1/2\sigma^2\right)\Delta t\right)\eta_1} \Phi\left(\frac{x-(\mu-1/2\sigma_d^2)\Delta t + \eta_1\sigma_d^2\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ &+ q\eta_2 \mathrm{e}^{1/2\eta_2^2\sigma^2\Delta t} \mathrm{e}^{\left(x-\left(\mu-1/2\sigma^2\right)\Delta t\right)\eta_2} \Phi\left(\frac{x-(\mu-1/2\sigma_d^2)\Delta t + \eta_2\sigma_d^2\Delta t}{\sigma\sqrt{\Delta t}}\right) \end{aligned} \tag{3.107}$$

CHAPTER FOUR

RESULTS

4.1 Preamble

The main focus was to develop models that best fit the dynamics of stock price processes obtained from the stock markets when jumps are detected.

In this chapter, the results for detecting jumps via models from the asymptotic theories of particular cases of the realised multipower variation (RMPV) process in the Nigerian All Shares Index (NASI), Japan stock Indices, and the UK stock Indices data were obtained. These results comprised the asymptotic theories for particular cases of the RMPV process, the convergence in distribution (law) of the difference of the realised variance (RV) and particular cases of the RMPV process. To achieve the aim, the asymptotic variances of the particular cases that is, $\varphi_{RBV}, \varphi_{RTV}, \varphi_{QPV}, \varphi_{PPV}, \varphi_{HPV}, \varphi_{H_PPV}, \varphi_{OpV}, \varphi_{NPV}$ *and* φ_{DPV} respectively for the realised bipower, tripower, quadpower, pentpower, hexpower, heptpower, octpower, nonpower and decpower variation processes were calculated. Then, the jump test models from the asymptotic properties of the particular cases of the RMPV process were also developed; these results were the extensions of the results in Barndorff-Nielsen and Shephard (2006), Barndorff-Nielsen *et al.* (2006a) and Ysusi (2006). The results enabled us determine more robust models than the existing BNS jump test method in literature. It was later established that the empirical data sets of the stock indices process present jumps, as could be seen in the plots of the stock indices processes.

Most of the existing jump-diffusion models in literature are driven by their exact analytical solutions, useful for Option Pricing. However, the motivation here was based not only on exact or closed-form solutions, but also on the consistency or compliance of models to the market price process. Hence, a family of skewed jump-diffusion models, with non-zero location parameters and scale parameters for upward and downward measures of the random jump processes was considered, for the dynamics of the stock's log returns price process. Therefore, new models that belong to the family of the skewed jump-diffusion models, namely: the *Asymmetric Laplace jump-diffusion (ALJD)* and the *modified double Rayleigh jump-diffusion (MDRJD)* models for the stock price process were proposed. The upward and downward random jump processes' measures were assumed to obey the asymmetric Laplace distribution and the modified double Rayleigh distribution with probabilities: p and q, where $p, q \geq 0; p + q = 1$. The probability density functions were derived via the convolution of densities for the log returns dynamics, and the Lévy-Khintchine formulae were obtained for the models, which were useful for the computation of moments of the processes. Also, given the log return processes' probability density functions, the optimal values of the parameters in the models were determined via the maximum likelihood estimation method.

Furthermore, owing to different values of the jump threshold, and by comparing the models' simulated densities with the densities of the empirical log returns of the stock indices data, a goodness of fit test to determine the best fit to the market was carried out. These models were compared with the Geometric Brownian Motion model in Black and Scholes (1973a), the symmetric Jump diffusion model (Merton's model) in Merton (1976) and Hanson and Westman (2002), and the asymmetric jump-diffusion model (Kou's model) in Kou (2002) to ascertain their compatibility with the dynamics of the stock price indices process obtained from the market.

4.2 STUDY ONE

Asymptotic properties of particular higher-order cases of the realised multipower variation process

The asymptotic properties of the realised multipower variation process in its generalised form were extensively discussed in chapter three in this thesis. Let the log return process X_t be in the class of the continuous stochastic volatility model ($Svsm^c$), then as $\Delta \to 0$

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{2/m}^{-m}\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)} - \{X\}_{\Delta,t}^{(2)}\right)$$

tends in law to $\mathrm{N}(0, \varphi_{RMPV})$

where,

$$\varphi_{RMPV} = Var\Big(|\nu|^2\Big) + \mu_{2/m}^{-2/m}\omega_m^2 - 2m\mu_{2/m}^{-m}Cov\Big(|\nu|^2, \prod_{i=0}^{m}|\nu_i|^{2/m}\Big) \tag{4.1}$$

and

$$\mu_{r_i} = \mathbb{E}(|\nu_i|^2) = \frac{2^{r/2}\Gamma(r/2+1/2)}{\sqrt{\pi}}, \nu \sim N(0,1) \tag{4.2}$$

4.2.1 The bipower variation process

The bipower variation process is the first order of the particular case of the generalised process, whose asymptotic distribution is given in equation (4.1), that is, for $m = 2$. That is, the particular case of the following convergence result was considered:

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RBV}) \tag{4.3}$$

In equation (4.3), the value $\varphi_{RMPV}(r_1, r_2, 2, \nu_i)$ is the **asymptotic variance** of the realised bipower variation process given as:

$$\varphi_{RBV} = \hat{\gamma}_1 + \mu_1^{-4}\omega_2^2 - 2\mu_1^{-2}\hat{\gamma}_2 \tag{4.4}$$

where,

$$\hat{\gamma}_1 = Var\Big(|\nu|^2\Big)$$
$$\hat{\gamma}_2 = 2Cov\Big(|\nu|^2, |\nu_1||\nu_2|\Big)$$
$$\omega_2^2 = Var\Big(|\nu_1||\nu_2|\Big) + 2Cov\Big(|\nu_1||\nu_2|, |\nu_2||\nu_3|\Big)$$

Given that ν_is are independent, $\nu_i \sim \mathrm{N}(0,1)$, and $\mu_r = \mathbb{E}(|\nu|^r) = \frac{2^{r/2}\Gamma(r/2+1/2)}{\sqrt{\pi}}$
For $Var(A) = E(A^2) - (E(A))^2$, $Cov(A,B) = E(A \cdot B) - E(A)E(B)$ and $\mu_2 = 1$, Then,

$$\begin{aligned} \hat{\gamma}_1 = Var\Big(|\nu|^2\Big) = 2 \\ \hat{\gamma}_2 = 2Cov\Big(|\nu|^2, |\nu_1||\nu_2|\Big) = 2\Big(\mu_3\mu_1 - \mu_2\mu_1^2\Big) \\ \omega_2^2 = Var\Big(|\nu_1||\nu_2|\Big) + 2Cov\Big(|\nu_1||\nu_2|, |\nu_2||\nu_3|\Big) \\ \omega_2^2 = 1 - 3\mu_1^4 + 2\mu_1^2 \end{aligned} \tag{4.5}$$

Thus, from equation (4.4),

$$\varphi_{RBV} = \mu_1^{-4} + 2\mu_1^{-2} - 5 \approx 0.60907 \tag{4.6}$$

4.2.2 The tripower variation process

Given the tripower variation process, the convergence result of the difference of the realised tripower variation and the realised variance is given as:

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{2/3}^{-3}\{X\}_{\Delta,t}^{(2/3,2/3,2/3)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RTV}) \tag{4.7}$$

where, φ_{RTV} is the asymptotic variance of the realised tripower variation process defined as:

$$\varphi_{RTV} = \hat{\gamma}_1 + \mu_{2/3}^{-6}\omega_3^2 - \Big(\mu_{2/3}^{-3}\hat{\gamma}_2 + \mu_{2/3}^{-3}\hat{\beta}_2\Big)$$

Since, $\hat{\gamma}_2 = \hat{\beta}_2$, Then,

$$\varphi_{RTV} = \hat{\gamma}_1 + \mu_{2/3}^{-6}\omega_3^2 - 2\mu_{2/3}^{-3}\hat{\gamma}_2 \tag{4.8}$$

where,

$$\hat{\gamma}_2 = 3Cov\Big(\nu^2, \nu_1^{2/3}\nu_2^{2/3}\nu_3^{2/3}\Big)$$

$$\omega_3^2 = Var\Big(\prod_{i=1}^{3}|\nu_i|^{2/3}\Big) + 2Cov\Big(\prod_{i=1}^{3}|\nu_i|^{2/3}, \prod_{i=2}^{4}|\nu_i|^{2/3}\Big) + 2Cov\Big(\prod_{i=1}^{3}|\nu_i|^{2/3}, \prod_{i=3}^{5}|\nu_i|^{2/3}\Big)$$

Calculating $\hat{\gamma}_2$ and ω_3^2, gives:

$$\hat{\gamma}_2 = 3Cov\Big(\nu^2, \prod_{i=1}^{m}|\nu_i|^{2/3}\Big) = 3\Big(\mu_{8/3}\mu_{2/3}^2 - \mu_{2/3}^3\Big)$$

$$\omega_3^2 = \mu_{4/3}^3 - \mu_{2/3}^6 + 2(\mu_{4/3}^2\mu_{2/3}^2 - \mu_{2/3}^6) + 2(\mu_{4/3}\mu_{2/3}^4 - \mu_{2/3}^6)$$

Thus, from equation (4.8),

$$\varphi_{RTV} = \mu_{4/3}\mu_{2/3}^{-2}\Big(\mu_{4/3}^2\mu_{2/3}^{-4} + 2\mu_{4/3}\mu_{2/3}^{-2} + 2\Big) - 7 \approx 1.0613 \tag{4.9}$$

4.2.3 The quadpower variation process

The convergence in distribution (law) result for the difference of the QPV and RV process for $r_1 = r_2 = r_3 = r_4 = 1/2$ and $m = 4$ is obtained as:

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{1/2}^{-4}\{X\}_{\Delta,t}^{(1/2,1/2,1/2,1/2)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RQV}) \tag{4.10}$$

where,

$$\varphi_{RQV} = \hat{\gamma}_1 + \mu_{1/2}^{-8}\omega_4^2 - 2\mu_{1/2}^{-4}\hat{\gamma}_2 \tag{4.11}$$

Thus,

$$\hat{\gamma}_2 = 4Cov\Big(\nu^2, \prod_{i=1}^{4}|\nu_i|^{1/2}\Big) \tag{4.12}$$

$$\begin{aligned}\omega_4^2 = Var\Big(\prod_{i=1}^{4}|\nu_i|^{1/2}\Big) &+ 2Cov\Big(\prod_{i=1}^{4}|\nu_i|^{1/4}, \prod_{i=2}^{5}|\nu_i|^{1/2}\Big)\\ &+ 2Cov\Big(\prod_{i=1}^{4}|\nu_i|^{1/2}, \prod_{i=3}^{6}|\nu_i|^{1/2}\Big)\\ &+ 2Cov\Big(\prod_{i=1}^{4}|\nu_i|^{1/2}, \prod_{i=4}^{7}|\nu_i|^{1/2}\Big)\end{aligned} \tag{4.13}$$

Calculating $\hat{\gamma}_2$ and ω_4^2, gives:

$$\hat{\gamma}_2 = 4Cov\Big(\mu_{5/2}\mu_{1/2}^3 - \mu_{1/2}^4\Big)$$

$$\omega_4^2 = \mu_1^4 - \mu_{1/2}^8 + 2(\mu_1^3\mu_{1/2}^2 - \mu_{1/2}^8) + 2(\mu_1^2\mu_{1/2}^4 - \mu_{1/2}^8) + 2(\mu_1\mu_{1/2}^6 - \mu_{1/2}^8$$

Hence, equation (4.11) becomes:

$$\varphi_{RQV} = \mu_1\mu_{1/2}^{-2}\Big(\mu_1^3\mu_{1/2}^{-6} + 2\mu_1^2\mu_{1/2}^{-4} + 2\mu_1\mu_{1/2}^{-2} + 2\Big) - 9 \approx 1.37702 \tag{4.14}$$

4.2.4 The pentpower variation process

Definition 4.1: The pentpower variation process

1. **Realised pentpower variation Process:** The realised pentpower variation process (RPPV) is an estimator of the pentpower process (RPPV) and is defined as:

$$\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_5)} = \Delta^{1-\delta(r_1,\ldots,r_5)} \sum_{j=1}^{c(t,5,\Delta)} f(x_j, r_i).$$

where, $f(x_j, r_i) = \prod_{i=0}^{4} \left|x_{j+i}\right|^{r_{i+1}}$

2. **Pentpower variation** (PPV) process is defined as the limit of the realised pentpower variation.

$$\{X\}_t^{(r_1,\ldots,r_5)} = \mathbb{P} - \lim_{\Delta\to 0}\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_5)} \tag{4.15}$$

where, $\mathbb{P} - \lim_{\Delta\to 0}$ denotes probability limit of the sum.

The realised pentpower variation process RPV is a particular case of RMPV processes. The convergence result of the difference between the RPV and RV process is obtained from the generalised form in equation (4.1) as: $(for\ r_i = 2/5, m = 5)$.

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{2/5}^{-5}\{X\}_{\Delta,t}^{(2/5,2/5,2/5,2/5,2/5)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RPV}) \tag{4.16}$$

where,

$$\varphi_{RPV} = \hat{\gamma}_1 + \mu_{2/5}^{10}\omega_5^2 - 2\mu_{2/5}^{-5}\hat{\gamma}_2 \tag{4.17}$$

and

$$\hat{\gamma}_2 = 5Cov\Big(|\nu_i^2|, \prod_{i=1}^{5} |\nu_i|^{2/5}\Big)$$

$$\begin{aligned}\omega_5^2 = Var\Big(\prod_{i=1}^{5} |\nu_i|^{2/5}\Big) &+ 2Cov\Big(\prod_{i=1}^{5} |\nu_i|^{2/5}, \prod_{i=2}^{6} |\nu_i|^{2/5}\Big)\\ &+2Cov\Big(\prod_{i=1}^{5} |\nu_i|^{2/5}, \prod_{i=3}^{7} |\nu_i|^{2/5}\Big)\\ &+2Cov\Big(\prod_{i=1}^{5} |\nu_i|^{2/5}, \prod_{i=4}^{8} |\nu_i|^{2/5}\Big)\\ &+2Cov\Big(\prod_{i=1}^{5} |\nu_i|^{2/5}, \prod_{i=5}^{9} |\nu_i|^{2/5}\Big)\end{aligned}$$

The values of $\hat{\gamma}_2$ *and* ω_5^2 were calculated to obtain:

$$\begin{aligned}\hat{\gamma}_2 &= 5\Big(\mu_{12/5}\mu_{2/5}^4 - \mu_{2/5}^5\Big) \; for \; (\mu_2 = 1)\\ \omega_5^2 &= \mu_{4/5}^5 - \mu_{2/5}^{10} + 2(\mu_{4/5}^4\mu_{2/5}^2\\ &\quad - \mu_{2/5}^{10}) + 2(\mu_{4/5}^3\mu_{2/5}^4 - \mu_{2/5}^{10}) + 2(\mu_{4/5}^2\mu_{2/5}^6\\ &\quad - \mu_{2/6}^{10}) + 2(\mu_{4/5}\mu_{2/5}^8 - \mu_{2/5}^{10})\end{aligned}$$

Therefore, equation (4.17) becomes:

$$\varphi_{RPV} = \mu_{4/5}\mu_{2/5}^{-2}\Big(\mu_{4/5}^4\mu_{2/5}^{-8} + 2\mu_{4/5}^3\mu_{2/5}^{-6} + 2\mu_{4/5}^2\mu_{2/5}^{-4} + 2\mu_{4/5}\mu_{2/5}^{-2} + 2\Big) - 11 \approx 1.60534 \tag{4.18}$$

4.2.5 The hexpower variation process

Definition 4.2: The hexpower variation process

1. **Realised Hexpower Variation Process:** The realised Hexpower variation process (RH_XV) is an estimator of the Hexpower process (H_XV) and is defined as:

$$\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_6)} = \Delta^{1-\delta(r_1,\ldots,r_6)} \sum_{j=1}^{c(t,6,\Delta)} f(x_j, r_i).$$

where, $f(x_j, r_i) = \prod_{i=0}^{5} \left|x_{j+i}\right|^{r_{i+1}}$

2. **Hexpower variation** (H_XV) process is defined as the limit of the realised Hexpower variation.

$$\{X\}_t^{(r_1,\ldots,r_6)} = \mathbb{P} - \lim_{\Delta \to 0} \{X\}_{\Delta,t}^{[r_1,r_2,\ldots,r_6]} \tag{4.19}$$

where, $\mathbb{P} - \lim_{\Delta\to 0}$ denotes probability limit of the sum.

4.2.5.1 Asymptotic property of the realised hexpower variation process

The convergence in law result for a particular case of the RH_xV process was obtained here, for $r_1 = r_2 = \ldots = r_6 = 1/3$ and for $m = 6$.
Thus,

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}} \left(\mu_{1/3}^{-6} \{X\}_{\Delta,t}^{[1/3,1/3,1/3,1/3,1/3,1/3]} - \{X\}_{\Delta,t}^{(2)} \right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RH_XV}) \tag{4.20}$$

where, φ_{RH_XV} is the asymptotic variance of the convergence result of the difference between the RH_XV and the RV processes.
Thus,

$$\varphi_{RH_XV} = \hat{\gamma}_1 + \mu_{1/3}^{-12}\omega_6^2 - 2\mu_{1/3}^{-6}\hat{\gamma}_2 \tag{4.21}$$

where,

$$\hat{\gamma}_2 = 6Cov\Big(|\nu_i^2|, \prod_{i=1}^{6} |\nu_i|^{1/3}\Big)$$

$$\begin{aligned}
\omega_6^2 = Var\Big(\prod_{i=1}^{6} |\nu_i|^{1/3}\Big) &+ 2Cov\Big(\prod_{i=1}^{6} |\nu_i|^{1/3}, \prod_{i=2}^{7} |\nu_i|^{1/3}\Big) \\
&+2Cov\Big(\prod_{i=1}^{6} |\nu_i|^{1/3}, \prod_{i=3}^{8} |\nu_i|^{1/3}\Big) \\
&+2Cov\Big(\prod_{i=1}^{6} |\nu_i|^{1/3}, \prod_{i=4}^{9} |\nu_i|^{1/3}\Big) \\
&+2Cov\Big(\prod_{i=1}^{6} |\nu_i|^{1/3}, \prod_{i=5}^{10} |\nu_i|^{1/3}\Big) \\
&+2Cov|\Big(\prod_{i=1} |\nu_1|^{1/3}, \prod_{i=6}^{11} |\nu_i|^{1/3}\Big)
\end{aligned}$$

Upon solving, the values for $\hat{\gamma}_2$ and ω_6^2 were obtained as:

$$\begin{aligned}
\hat{\gamma}_2 &= 6\Big(\mu_{7/3}\mu_{1/3}^5 - \mu_{1/3}^6\Big)\ for\ (\mu_2 = 1) \\
\omega_6^2 &= \mu_{2/3}^6 - \mu_{1/3}^{12} + 2(\mu_{2/3}^5\mu_{1/3}^2 \\
&\quad - \mu_{1/3}^{12}) + 2(\mu_{2/3}^4\mu_{1/3}^4 - \mu_{1/3}^{12}) + 2(\mu_{2/3}^3\mu_{1/3}^6 \\
&\quad - \mu_{1/3}^{112}) + 2(\mu_{2/3}^2\mu_{1/3}^8 - \mu_{1/3}^{12}) \\
&\quad + 2(\mu_{2/3}\mu_{1/3}^{10} - \mu_{1/3}^{12})
\end{aligned}$$

Thus, substituting into equation (4.21) it follows that:

$$\begin{aligned}
\varphi_{RH_XV} &= \mu_{2/3}\mu_{1/3}^{-2}\Big(\mu_{2/3}^5\mu_{1/3}^{-10} + 2\mu_{2/3}^4\mu_{1/3}^{-8} + 2\mu_{2/3}^3\mu_{1/3}^{-6} \\
&\quad + 2\mu_{2/3}^2\mu_{1/3}^{-4} + 2\mu_{2/3}\mu_{1/3}^{-2} + 2\Big) - 13 \approx 1.776889
\end{aligned} \tag{4.22}$$

4.2.6 The Heptpower variation process

Definition 4.3: The Heptpower variation process

1. **Realised Heptpower Variation Process:**. The realised Heptpower variation process (RH_pV) is an estimator of the Heptpower variation (H_pV) process and is defined as:

$$\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_7)} = \Delta^{1-\delta(r_1,\ldots,r_7)} \sum_{j=1}^{c(t,7,\Delta)} f(x_j, r_i).$$

where, $f(x_j, r_i) = \prod_{i=0}^{6} \left|x_{j+i}\right|^{r_{i+1}}$

2. **The Heptpower variation** process is defined as the limit of the (RH_pV) process given as:

$$\{X\}_t^{(r_1,\ldots,r_7)} = \mathbb{P} - \lim_{\Delta\to 0}\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_7)} \tag{4.23}$$

4.2.6.1 Asymptotic properties of the realised heptpower variation process

With reference to theorem (3.11), and for the RH_pV process,

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{2/7}^{-7}\{X\}_{\Delta,t}^{(2/7,2/7,2/7,2/7,2/7,2/7,2/7)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RH_pV}) \qquad (4.24)$$

where, φ_{RH_pV} is the asymptotic variance of the convergence result of the RH_pV process. Now, the value of the φ_{RH_pV} was obtained as follows:

$$\varphi_{RH_XV} = \hat{\gamma}_1 + \mu_{2/7}^{-14}\omega_7^2 - 2\mu_{2/7}^{-7}\hat{\gamma}_2 \qquad (4.25)$$

where,

$$\hat{\gamma}_2 = 7Cov\Big(|\nu_i^2|, \prod_{i=1}^{7}|\nu_i|^{2/7}\Big)$$

$$\begin{aligned}
\omega_7^2 = Var\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}\Big) &+ 2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=2}^{8}|\nu_i|^{2/7}\Big)\\
&+2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=3}^{9}|\nu_i|^{2/7}\Big)\\
&+2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=4}^{10}|\nu_i|^{2/7}\Big)\\
&+2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=5}^{11}|\nu_i|^{2/7}\Big)\\
&+2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=6}^{12}|\nu_i|^{2/7}\Big)\\
&+2Cov\Big(\prod_{i=1}^{7}|\nu_i|^{2/7}, \prod_{i=1}^{13}|\nu_i|^{2/7}\Big)
\end{aligned}$$

To obtain φ_{RH_pV} in equation (4.25), the values of $\hat{\gamma}_1, \hat{\gamma}_2,\ and\ \omega_7^2$ were computed to get:

$$\hat{\gamma}_1 = 2$$
$$\hat{\gamma}_2 = 7\Big(\mu_{16/7}\mu_{6/7}^6 - \mu_{2/7}^7\Big)\ for\ (\mu_2 = 1)$$
$$\omega_7^2 = \mu_{4/7}^7 - \mu_{2/7}^{14} + 2(\mu_{4/7}^6\mu_{2/7}^2 - \mu_{2/7}^{14}) + 2(\mu_{4/7}^5\mu_{2/7}^4 - \mu_{2/7}^{14}) + 2(\mu_{4/7}^4\mu_{2/7}^6 - \mu_{2/7}^{14}) + 2(\mu_{4/7}^3\mu_{2/7}^8 - \mu_{2/7}^{14}) + 2(\mu_{4/7}^2\mu_{2/7}^{10} - \mu_{2/7}^{14}) + 2(\mu_{4/7}\mu_{2/7}^{12} - \mu_{2/7}^{14})$$

Thus, it follows from equation (4.25) that:

$$\varphi_{RH_pV} = \mu_{4/7}\mu_{2/7}^{-2}\Big(\mu_{4/7}^6\mu_{2/7}^{-12} + 2\mu_{4/7}^5\mu_{2/7}^{-10} + 2\mu_{4/7}^4\mu_{2/7}^{-8} + 2\mu_{4/7}^3\mu_{2/7}^{-6} + 2\mu_{4/7}^2\mu_{2/7}^{-4} + 2\mu_{4/7}\mu_{2/7}^{-2} + 2\Big) - 15 \approx 1.910028 \tag{4.26}$$

4.2.7 The octpower variation process

Definition 4.4: The octpower variation process

1. The **realised octpower variation** (RO_pV) process is an estimator of the octpower variation (O_pV) process and its defined as:

$$\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_8)} = \Delta^{1-\delta(r_1,\ldots,r_8)} \sum_{j=1}^{c(t,8,\Delta)} f(x_j, r_i).$$

where, $f(x_j, r_i) = \prod_{i=0}^{7} \left|x_{j+i}\right|^{r_{i+1}}$

2. The **octpower Variation** (O_pV) process is defined as the probability limit of the (RO_pV) process given as:

$$\{X\}_t^{[r_1,\ldots,r_8]} = \mathbb{P} - \lim_{\Delta\to 0}\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_8)} \tag{4.27}$$

where, $\mathbb{P} - \lim(.)$ is as defined above.

4.2.7.1 Asymptotic property of the octpower variation Process

Now, it follows from theorem (3.11), that the asymptotic property of the particular case of the RO_pV process, that is, for $m = 8$, and $r_i = 1/4, i = 1, \ldots, 8$, gives:

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{1/4}^{-8}\{X\}_{\Delta,t}^{(1/4,1/4,1/4,1/4,1/4,1/4,1/4,1/4)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RO_pV}) \quad (4.28)$$

where, φ_{RO_pV} is the asymptotic variance of the convergence result of the difference between the RO_pV and the RV processes.
Thus,

$$\varphi_{RO_pV} = \hat{\gamma}_1 + \mu_{1/4}^{-16}\omega_8^2 - 2\mu_{1/4}^{-8}\hat{\gamma}_2 \quad (4.29)$$

The value of φ_{RO_pV}, given above, was calculated by finding the values of $\hat{\gamma}_2$ and ω_8^2.

$$\hat{\gamma}_2 = 8Cov\Big(|\nu_i^2|, \prod_{i=1}^{8}|\nu_i|^{2/7}\Big)$$

$$\begin{aligned}
\omega_8^2 = Var\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}\Big) &+ 2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=2}^{9}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=3}^{10}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=4}^{11}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=5}^{12}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=6}^{13}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=7}^{14}|\nu_i|^{1/4}\Big) \\
&+2Cov\Big(\prod_{i=1}^{8}|\nu_i|^{1/4}, \prod_{i=8}^{15}|\nu_i|^{1/4}\Big)
\end{aligned}$$

Then, the values of $\hat{\gamma}_1, \hat{\gamma}_2,\ and\ \omega_8^2$ were obtained as follows:

$$
\begin{aligned}
\hat{\gamma}_1 &= Var(|\nu|^2) = 2\\
\hat{\gamma}_2 &= 8\Big(\mu_{9/4}\mu_{1/4}^{7} - \mu_{1/4}^{8}\Big)\\
\omega_8^2 &= \mu_{1/2}^{8} - \mu_{1/4}^{16} + 2(\mu_{1/2}^{7}\mu_{1/4}^{2} - \mu_{1/4}^{16})\\
&\quad + 2(\mu_{1/2}^{6}\mu_{1/4}^{4} - \mu_{1/4}^{16}) + 2(\mu_{1/2}^{5}\mu_{1/4}^{6} - \mu_{1/4}^{16})\\
&\quad + 2(\mu_{1/2}^{4}\mu_{1/4}^{8} - \mu_{1/4}^{16}) + 2(\mu_{1/2}^{3}\mu_{1/4}^{10} - \mu_{1/4}^{16})\\
&\quad + 2(\mu_{1/2}^{2}\mu_{1/4}^{12} - \mu_{1/4}^{16}) + 2(\mu_{1/2}\mu_{1/4}^{14} - \mu_{1/4}^{16}
\end{aligned}
$$

Thus, from equation (4.29) the value of φ_{RO_pV} was obtained as follows:

$$
\begin{aligned}
\varphi_{RO_pV} &= \mu_{1/2}\mu_{1/4}^{-2}\Big(\mu_{1/2}^{7}\mu_{1/4}^{-14} + 2\mu_{1/2}^{6}\mu_{1/4}^{-12}\\
&\quad + 2\mu_{1/2}^{5}\mu_{1/4}^{-10} + 2\mu_{1/2}^{4}\mu_{1/4}^{-8} + 2\mu_{1/2}^{3}\mu_{1/4}^{-6} + 2\mu_{1/2}^{2}\mu_{1/4}^{-4}\\
&\quad + 2\mu_{1/2}\mu_{1/4}^{-2} + 2\Big) - 17 \approx 2.016148
\end{aligned} \tag{4.30}
$$

Therefore, equation (4.30) gives an expression for φ_{RO_pV} in terms of μ_{r_i}.

4.2.8 The nonpower variation process

Definition 4.5: The nonpower variation process

1. The **realised Nonpower** variation (RNV) process is an estimator of the Nonpower variation NV process and its defined as:

$$
\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_9)} = \Delta^{1-\delta(r_1,\ldots,r_9)} \sum_{j=1}^{c(t,9,\Delta)} f(x_j, r_i).
$$

where, $f(x_j, r_i) = \prod_{i=0}^{8} \left|x_{j+i}\right|^{r_{i+1}}$

2. The **nonpower variation** (NV) process is defined as the probability limit of the (RNV) process and given as:

$$
\{X\}_{t}^{(r_1,\ldots,r_9)} = \mathbb{P} - \lim_{\Delta\to 0}\{X\}_{\Delta,t}^{[r_1,r_2,\ldots,r_9]} \tag{4.31}
$$

where, $\mathbb{P} - \lim(.)$ is as defined above.

4.2.8.1 Asymptotic property of the nonpower variation process

The asymptotic property of the difference of the RV process and the RNV process, that is, for $m = 9$, and $r_i = 2/9, i = 1, \ldots, 9$ was obtained in this subsection.

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}} \left(\mu_{2/9}^{-9} \{X\}_{\Delta,t}^{(2/9,2/9,2/9,2/9,2/9,2/9,2/9,2/9,2/9)} - \{X\}_{\Delta,t}^{(2)} \right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RNV}) \tag{4.32}$$

where, φ_{RNV} is the asymptotic variance of the convergence in law of the difference of the RNV and RV processes which is obtained as:

$$\varphi_{RNV} = \hat{\gamma}_1 + \mu_{2/9}^{-18} \omega_9^2 - 2\mu_{2/9}^{-9} \hat{\gamma}_2 \tag{4.33}$$

where,

$$\hat{\gamma}_2 = 9Cov\Big(|\nu_i^2|, \prod_{i=1}^{9} |\nu_i|^{2/9} \Big)$$

and

$$\begin{aligned}
\omega_9^2 = Var\Big(\prod_{i=1}^{9} |\nu_i|^{2/9} \Big) &+ 2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=2}^{10} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=3}^{11} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=4}^{12} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=5}^{13} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=6}^{14} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=7}^{15} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=8}^{16} |\nu_i|^{2/9} \Big) \\
&+2Cov\Big(\prod_{i=1}^{9} |\nu_i|^{2/9}, \prod_{i=9}^{17} |\nu_i|^{2/9} \Big)
\end{aligned}$$

4.2.8.2 The asymptotic variance in terms of μ_r

Here, the asymptotic variance φ_{RNPV} of the realised nonpower variation (RNPV) process given its expression in terms of μ_r's was calculated. Recall that:

$$\mu_r = \frac{2^{r/2}\Gamma(r/2+1/2)}{\sqrt{\pi}} \tag{4.34}$$

The values of $\hat{\gamma}_1, \hat{\gamma}_2$ and ω_9^2 in terms of μ_r were obtained.

Therefore,

$$\begin{aligned}
\hat{\gamma}_1 &= 2 \\
\hat{\gamma}_2 &= 9\Big(\mu_{20/9}\mu_{2/9}^{8} - \mu_{2/9}^{9}\Big) \\
\omega_9^2 &= \mu_{4/9}^{9} - \mu_{2/9}^{18} + 2(\mu_{4/9}^{8}\mu_{2/9}^{2} - \mu_{2/9}^{18}) \\
&\quad + 2(\mu_{4/9}^{7}\mu_{2/9}^{4} - \mu_{2/9}^{18}) + 2(\mu_{4/9}^{6}\mu_{2/9}^{6} - \mu_{2/9}^{18}) \\
&\quad + 2(\mu_{4/9}^{5}\mu_{2/9}^{8} - \mu_{2/9}^{18}) + 2(\mu_{4/9}^{4}\mu_{2/9}^{10} - \mu_{2/9}^{18}) \\
&\quad + 2(\mu_{4/9}^{3}\mu_{2/9}^{12} - \mu_{2/9}^{18}) + 2(\mu_{4/9}^{2}\mu_{2/9}^{14} - \mu_{2/9}^{18} \\
&\quad + 2(\mu_{4/9}\mu_{2/9}^{16} - \mu_{2/9}^{18})
\end{aligned}$$

Substituting the above expression for $\hat{\gamma}_1, \hat{\gamma}_2$ and ω_9^2 into equation (4.33) gives:

$$\begin{aligned}
\varphi_{RNV} &= \mu_{4/9}\mu_{2/9}^{-2}\Big(\mu_{4/9}^{8}\mu_{2/9}^{-16} + 2\mu_{4/9}^{7}\mu_{2/9}^{-14} \\
&\quad + 2\mu_{4/9}^{6}\mu_{2/9}^{-12} + 2\mu_{4/9}^{5}\mu_{2/9}^{-10} + 2\mu_{4/9}^{4}\mu_{2/9}^{-8} + 2\mu_{4/9}^{3}\mu_{2/9}^{-6} \\
&\quad + 2\mu_{4/9}^{2}\mu_{2/9}^{-4} + 2\mu_{4/9}\mu_{2/9}^{-2} + 2\Big) - 19 \approx 2.102613
\end{aligned} \tag{4.35}$$

Hence, equation (4.35) gives an expression for φ_{RNV} in terms of μ_{r_i}.

4.2.9 The decpower variation process

Definition 4.6: The decpower variation process

1. The Realised Decpower variation (RDV) process is an estimator of the Decpower variation DV process and defined as:

$$\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_m)} = \Delta^{1-\delta(r_1,\ldots,r_{10})} \sum_{j=1}^{c(t,10,\Delta)} f(x_j, r_i).$$

where, $f(x_j, r_i) = \prod_{i=0}^{9} \left|x_{j+i}\right|^{r_{i+1}}$

2. The decpower variation (DV) process is the probability limit of the (RDV) process and it's given as:

$$\{X\}_t^{(r_1,\ldots,r_{10})} = \mathbb{P} - \lim_{\Delta \to 0} \{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_m)} \tag{4.36}$$

where $\mathbb{P} - \lim(.)$ is as defined above.

4.2.9.1 Asymptotic property of the decpower variation Process

Here, the asymptotic property of the difference of the RV process and the RDV process was obtained. That is, for $m = 10$, and $r_i = 1/5, \quad i = 1, \ldots, 10$. Hence, were have that the convergence in distribution of a difference of the RV and the RDV processes is given for $r_i = 1/5, i = 1, \ldots, 10$ *and for* $m = 10$ as:

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}} \left(\mu_{1/5}^{-10} \{X\}_{\Delta,t}^{[1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5]} - \{X\}_{\Delta,t}^{(2)} \right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RDV}) \tag{4.37}$$

where, φ_{RDV} is the asymptotic variance of the convergence in law of the difference of the RDV and RV processes which is obtained as:

$$\varphi_{RDV} = \hat{\gamma}_1 + \mu_{1/5}^{-20} \omega_{10}^2 - 2\mu_{1/5}^{-10} \hat{\gamma}_2 \tag{4.38}$$

where,

$$\hat{\gamma}_2 = 10Cov\Big(|\nu_i^2|, \prod_{i=1}^{10} |\nu_i|^{1/5} \Big)$$

$$
\begin{aligned}
\omega_{10}^2 = Var\Big(\prod_{i=1}^{10}|\nu_i|^{1/5}\Big) &+ 2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=2}^{11}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=3}^{12}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=4}^{13}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=5}^{14}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=6}^{15}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=7}^{16}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=8}^{17}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=9}^{18}|\nu_i|^{1/5}\Big)\\
&+2Cov\Big(\prod_{i=1}^{10}|\nu_i|^{1/5},\prod_{i=10}^{19}|\nu_i|^{1/5}\Big)
\end{aligned}
$$

4.2.9.2 The asymptotic variance of the realised decpower variation Process

The asymptotic variance φ_{RDV} of the realised Decpower variation (RDV) process in terms of μ_r's was obtained here. Recall that:

$$\mu_r = \frac{2^{r/2}\Gamma(r/2+1/2)}{\sqrt{\pi}} \tag{4.39}$$

In this case, $r_i = 1/5, i = 1, \ldots, 10$ *and* $\sum_{i=1}^{10} r_i = 2$ Thus, the values of $\hat{\gamma}_1, \hat{\gamma}_2$ *and* ω_{10}^2 in terms of μ_r were obtained.

Therefore,

$$\begin{aligned}
\hat{\gamma}_2 &= 10\Big(\mu_{11/5}\mu_{1/5}^{9} - \mu_{1/5}^{10}\Big) \\
\omega_{10}^2 &= \mu_{2/5}^{10} - \mu_{1/5}^{20} + 2(\mu_{2/5}^{9}\mu_{1/5}^{2} - \mu_{1/5}^{20}) \\
&\quad + 2(\mu_{2/5}^{8}\mu_{1/5}^{4} - \mu_{1/5}^{20}) + 2(\mu_{2/5}^{7}\mu_{1/5}^{6} - \mu_{1/5}^{20}) \\
&\quad + 2(\mu_{2/5}^{6}\mu_{1/5}^{8} - \mu_{1/5}^{20}) + 2(\mu_{2/5}^{5}\mu_{1/5}^{10} - \mu_{1/5}^{20}) \\
&\quad + 2(\mu_{2/5}^{4}\mu_{1/5}^{12} - \mu_{1/5}^{20}) + 2(\mu_{2/5}^{3}\mu_{1/5}^{14} - \mu_{1/5}^{18} \\
&\quad + 2(\mu_{2/5}^{2}\mu_{1/5}^{16} - \mu_{1/5}^{20}) + 2(\mu_{2/5}\mu_{1/5}^{18} - \mu_{1/5}^{20})
\end{aligned}$$

Substituting, for $\hat{\gamma}_1, \hat{\gamma}_2$ *and* ω_{10}^2 in equation (4.38), gives:

$$\begin{aligned}
\varphi_{RDV} &= \mu_{2/5}\mu_{1/5}^{-2}\Big(\mu_{2/5}^{9}\mu_{1/5}^{-18} + 2\mu_{2/5}^{8}\mu_{1/5}^{-16} \\
&+ 2\mu_{2/5}^{7}\mu_{1/5}^{-14} + 2\mu_{2/5}^{6}\mu_{1/5}^{-12} + 2\mu_{2/5}^{5}\mu_{1/5}^{-10} + 2\mu_{2/5}^{4}\mu_{1/5}^{-8} + 2\mu_{2/5}^{3}\mu_{1/5}^{-6} \\
&+ 2\mu_{2/5}^{2}\mu_{1/5}^{-4} + 2\mu_{2/5}\mu_{1/5}^{-2} + 2\Big) - 21 \approx 2.174364
\end{aligned} \tag{4.40}$$

Hence, equation (4.40) gives the value of φ_{RDV} in terms of μ_r.

4.3 STUDY TWO

The jump test models

This thesis considered the stock indices price process as a jump-diffusion model. As a result, the **presence** or **absence** of jumps in stock indices data was first investigated. This was achieved by applying jump test method derived from an extension of the BNS-jump test to the data. Under the null hypothesis, that the log returns process belongs to the $Svsm^c$ and under the alternative hypothesis, that it belongs to $Svsm^j$.

Given the above-mentioned, the jump test models for the particular cases of the RMPV processes from the asymptotic results obtained in section (4.1) of this thesis, were derived and applied to the stock indices data.

4.3.1 The RMPV jump test models

In the work of Barndorff-Nielsen and Shephard (2006), three types of test for jumps based on the (1, 1)-bipower variation process are observed. These are the linear jump test, the ratio jump test and adjusted ratio jump test. In the case of the generalised realised multipower variation (RMPV) process $\{X\}_{\Delta,t}^{(r_1,r_2,\ldots,r_m)}$, the linear jump test is based on the asymptotic properties discussed in section (4.1). That is,

$$\frac{\Delta^{-1/2}}{\sqrt{\int_0^t \sigma^4 ds}}\left(\mu_{2/m}^{-m}\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)} - \{X\}_{\Delta,t}^{(2)}\right) \xrightarrow{L} \mathrm{N}(0, \varphi_{RMPV}) \tag{4.41}$$

where, φ_{RMPV} is the asymptotic variance of the convergence in distribution of the difference of the RV and the $RMPV$ processes. Given that $\{X\}_{\Delta,t}^{(2)} \xrightarrow{L} \int_0^t \sigma^2 ds$, then, the ratio jump test was obtained from equation (4.41) by dividing through by $\{X\}_{\Delta,t}^{(2)} = \int_0^t \sigma^2 ds$. Hence,

$$\frac{\left(\mu_{2/3}^{m}\frac{\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\Delta^{1/2}\varphi_{RMPV}\sqrt{\frac{\int_0^t \sigma^4 ds}{(\int_0^t \sigma^2 ds)^2}}} \xrightarrow{L} \mathrm{N}(0,1) \tag{4.42}$$

In this work, real life empirical observations of the stock market indices were dealt with, which are not continuous in the actual sense, hence, the feasible adjusted ratio jump test for the particular cases of the model in equation (4.42) were obtained by the following steps:

1. The values of the component $\int_0^t \sigma_s^4 ds$ and $\int_0^t \sigma_s^2 ds$ cannot be observed when working with the discrete data, estimators for the quantities are needed.

2. Let $\hat{q}$ be the estimator for $\int_0^t \sigma_s^4 ds$ and $\hat{p}$ be the estimator for $\int_0^t \sigma_s^2 ds$ where, $\hat{p}$ is the realised quad power variation $\{X\}_{\Delta,t}^{[1,1,1,1]}$ process given as:

$$\hat{q} = \mu_1^{-4} \sum_{j=4}^{n} |x_j||x_{j+1}||x_{j+2}||x_{j+3}| = \mu_1^{-4} \sum_{j=4}^{n} \prod_{i=0}^{3} |x_{j+i}| \tag{4.43}$$

and the estimator for $\int_0^t \sigma_s^2 ds$ is the realised bipower variation process $\{X\}_{\Delta,t}^{(1,1)}, \hat{p}$ given as;

$$\hat{p} = \mu_1^{-2} \sum_{j=2}^{n} |x_j||x_{j+1}| = \mu_1^{-2} \{X\}_{\Delta,t}^{(1,1)} \tag{4.44}$$

3. Now, a feasible ratio jump test for the RMPV model was obtained and given as:

$$\frac{\left(\mu_{2/m}^{-m} \frac{\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{RMPV} \Delta^{1/2} \sqrt{\frac{\hat{q}}{\hat{p}^2}}} = \hat{\mathrm{z}} \tag{4.45}$$

where, $\hat{\mathrm{z}} \sim \mathrm{N}(0,1)$.

4. By Jensen's inequality, in the feasible test (Barndorff-Nielsen and Shephard, 2003b);

$$\frac{\hat{q}}{\hat{p}^2} \geq 1$$

Hence, the adjusted ratio for the estimators $\frac{\hat{q}}{\hat{p}^2}$ was employed to get:

$$\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)$$

Thus, the **feasible adjusted ratio jump test** for the RMPV model was obtained as follows:

$$\hat{\mathrm{z}}_{\mathrm{m}} = \frac{\left(\frac{\mu_{2/m}^{-m} \{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{RMPV} \Delta^{1/2} \sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.46}$$

Equation (4.46) is the jump test model that will be used in this work for particular cases, subject to the following hypotheses:

$$H_0 : X_t \in Svsm^c$$

.

$$H_1 : X_t \in Svsm^j$$

Thus, for the BP case, it followed from equation (4.46) that:

$$\hat{z}_2 = \frac{\left(\frac{\mu_1^{-2}\{X\}_{\Delta,t}^{(1,1)}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{BP}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.47}$$

where, $\varphi_{BP} = 0.6090$.

For the realised tripower variation process, were have that:

$$\hat{z}_3 = \frac{\left(\frac{\mu_{2/3}^{-3}\{X\}_{\Delta,t}^{(2/3,2/3,2/3)}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{TP}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.48}$$

where, $\varphi_{RTP} \approx 1.0613$

The RQV jump test model is given as:

$$\hat{Z}_4 = \frac{\left(\frac{\mu_{1/2}^{-4}\{X\}_{\Delta,t}^{[1/4,1/4,1/4,1/4]}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{RQV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.49}$$

where, $\varphi_{RQP} \approx 1.3770$

The RPV jump test model is given as:

$$\hat{z}_5 = \frac{\left(\frac{\mu_{2/5}^{-5}\{X\}_{\Delta,t}^{[2/5,2/5,2/5,2/5,2/5]}}{\{X\}_{\Delta,t}^{(2)}} - 1\right)}{\varphi_{RPV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.50}$$

where, $\varphi_{RPV} \approx= 1.6053$

The RH_XV jump test model is given as:

$$\hat{Z}_6 = \frac{\left(\frac{\mu_{1/3}^{-6}\{X\}_{\Delta,t}^{[1/3,1/3,1/3,1/3,1/3,1/3]}}{\{X\}_{\Delta,t}^{(2)}} - 1 \right)}{\varphi_{RH_XV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.51}$$

where, $\varphi_{RH_XV} \approx 1.7769$

The RH_pV jump test model is given as:

$$\hat{Z}_7 = \frac{\left(\frac{\mu_{2/7}^{-7}\{X\}_{\Delta,t}^{[2/7,2/7,2/7,2/7,2/7,2/7,2/7]}}{\{X\}_{\Delta,t}^{(2)}} - 1 \right)}{\varphi_{RH_pV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.52}$$

where, $\varphi_{RH_pV} \approx 1.9100$

The RO_pV jump test model is given as:

$$\hat{Z}_8 = \frac{\left(\frac{\mu_{1/4}^{-8}\{X\}_{\Delta,t}^{[1/4,1/4,1/4,1/4,1/4,1/4,1/4,1/4]}}{\{X\}_{\Delta,t}^{(2)}} - 1 \right)}{\varphi_{RO_pV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.53}$$

where, $\varphi_{RO_pV} \approx 2.0161$

The RNV jump test model is given as:

$$\hat{Z}_9 = \frac{\left(\frac{\mu_{2/9}^{-9}\{X\}_{\Delta,t}^{[2/9,2/9,2/9,2/9,2/9,2/9,2/9,2/9,2/9]}}{\{X\}_{\Delta,t}^{(2)}} - 1 \right)}{\varphi_{RNV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.54}$$

where, $\varphi_{RNV} \approx 2.1026$

The RDV jump test model is given as:

$$\hat{Z}_{10} = \frac{\left(\frac{\mu_{1/5}^{-10}\{X\}_{\Delta,t}^{[1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5,1/5]}}{\{X\}_{\Delta,t}^{(2)}} - 1 \right)}{\varphi_{RDV}\Delta^{1/2}\sqrt{\max\left(1, \frac{\hat{q}}{\hat{p}^2}\right)}} \tag{4.55}$$

where, $\varphi_{RDV} \approx 2.1744$

Hence, the particular higher-order cases of the RMPV processes have been used to obtain the jump test models for detecting the presence or absence of jumps in financial data under the assumption that the log returns obeys a continuous stochastic volatility semimartingale process. The models given in equations (4.47)-(4.55) were the jump test models for bipower, tripower, quadpower, pentpower, hexpower, heptpower, octpower, nonpower and decpower variation processes. It was noted that the asymptotic variances obtained in each case varied one to another.

4.4 STUDY THREE

Stock market indices realisation and description

The descriptions of the stock data's empirical features obtained from the Nigerian stock market, Japan stock market, and the UK stock market were given in this study. The three stock markets were selected based on the following reasons:

The first reason for performing the analysis for these selected countries stems from the available stock market price data in the exchanges located in these countries.

These countries have heterogeneous levels of developed exchanges reflected in the differences in market pricing mechanism, available financial instruments, and sophistication of trading strategies used by agents in the exchanges, all of which feed into the observed price in the market. This heterogeneity has been documented in literature, that is, comparing emerging markets vis-à-vis developed markets (see Dong *et al.*, 2020 and Jin *et al.*, 2020).

The heterogeneity between UK and Japan based on this dimension might be small, however, the flow of information that is time-varying can lead to the transmission of volatility between stock markets. This is not peculiar to these countries, but such observations between UK and Japan can be very peculiar due to differences in peak trading period (Tokyo is 8 hours ahead of London).

Although, one might argue that price shocks from published news should affect all markets at the same time, albeit in different ways because the significance of a piece of news may vary from country to country. Nevertheless, not all information, nor the ability to process it, is public. Valuable information is contained in the prices that other traders are willing to pay. Hence, individuals trading in London may feel that information is revealed by the price changes in the Tokyo stock exchanges. These observations of prices driven by micro-structural behaviours of different agents can lead to differences in jumps that occur at different times. This will most likely be the case in sophisticated exchanges characterised by different peak trading times.

The inclusion of Nigeria in the analysis alongside Japan and the UK helps us examine the differences in jumps between developed and emerging stock markets. While the behaviour of jumps between Japan and the UK helps us characterise the differences in jumps as time difference in the propagation of information shocks.

4.4.1 The Nigerian ASI data realisation and description

The Nigerian All Share Index (NASI) data was obtained from the Nigerian Stock Exchange (NSE). The sample period was twenty-two-years, from January 2, 1998, to February 21, 2020, which consist of 5334 trading days. The reason for choosing the All-share Indices (ASI) data was that, it gives a true picture of the movement of the price in the Nigerian stock market and depicts the general behaviour of all the shares traded on the Nigerian Stock Exchange daily. Since the NASI is short of intra-day data, the available NASI inter-day data (that is, one observation in a day) were obtained. In the sequel, vivid descriptions of the sample paths of the price process and log returns were given. Graphical descriptions of the ASI data and its log-returns were presented in Figures 4.1 and 4.2 below.

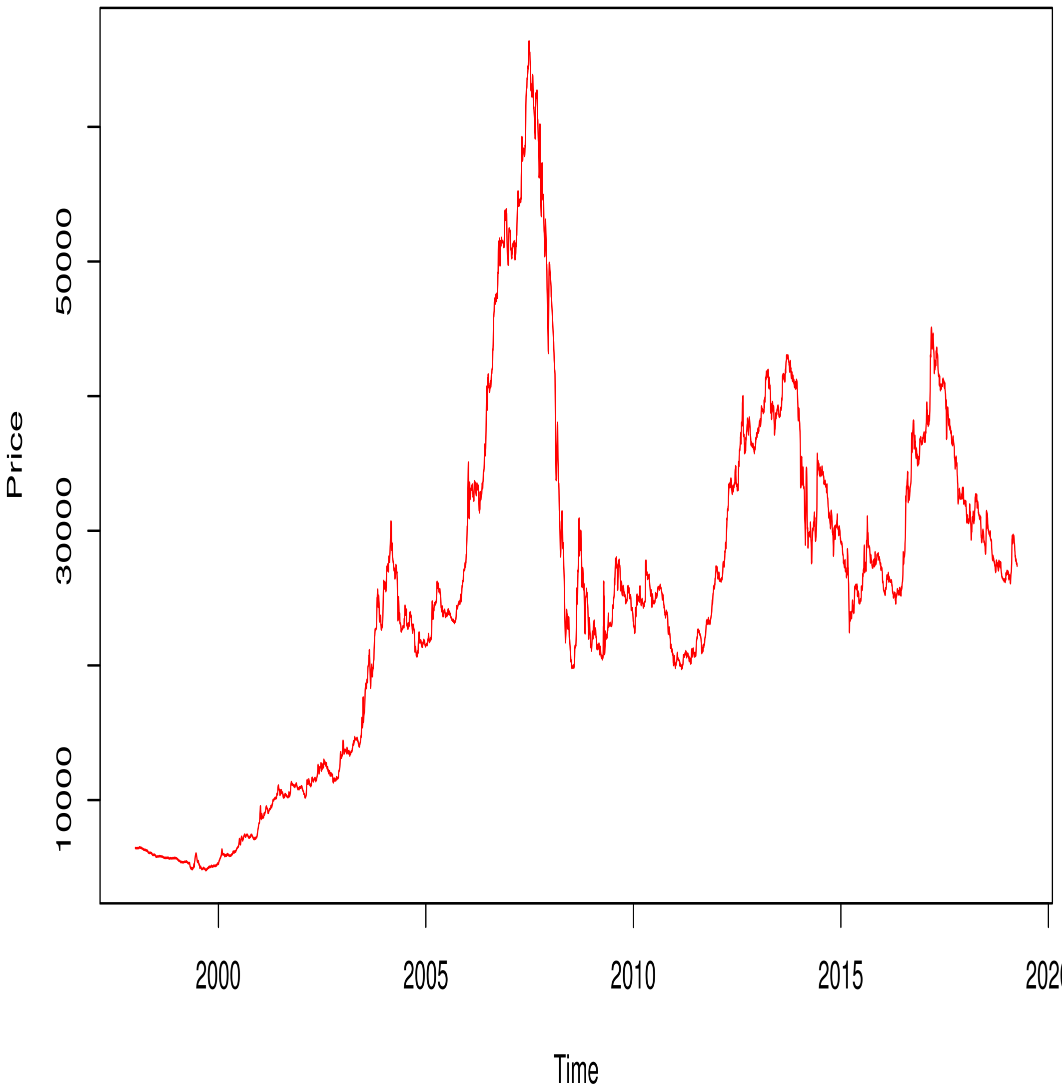

Figure 4.1: Price process ($\tilde{S}_t^{nig}$) of the Nigerian All Share Index from January 2, 1998, to February 21, 2020

Figure 4.1 shows the price process ($\tilde{S}_t^{nig}$) of the Nigerian All Share Index from January 2, 1998, to February 21, 2020, comprising 5334 daily observations of the All Share Indices (ASI). The values of the ASI were found to be relatively high in the year 2007 and low in the year 2000 and also display a high sense of variability. There was a drastic decrease in the value of $\tilde{S}_t^{nig}$ in 2008/2009, this could have been caused by economic meltdown effect in 2007, Owoloko and Okeke (2014) buttressed this fact too.

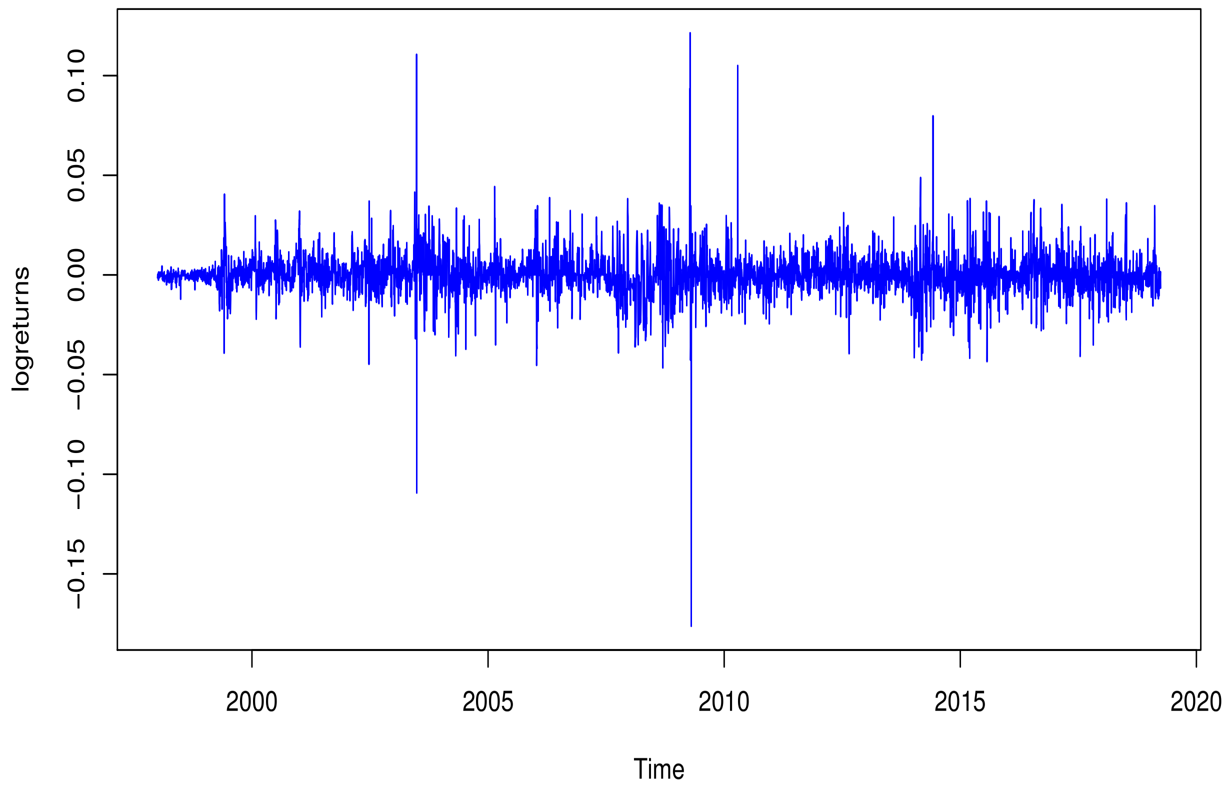

Figure 4.2: Log-returns $(ln\frac{S^{nig}_{t+\Delta t}}{\tilde{S}_t})$ of the Nigerian All Share Index

The plot of the log-returns of the Nigerian ASI was presented in Figure 4.2. It can be clearly seen from Figure 4.2 that there are small and large spikes of different sizes, indicating that jumps are likely to be present in the log-returns. This gives us the motivation to investigate empirically the absence or presence of jumps in the log-returns and also to vary the threshold of jumps in this thesis. The mean of the empirical log return of the NASI data is found to be: $\mathbb{E}^{(NASI)} = 2.7 \times 10^{-4}$, the variance is $\mathbb{V}^{(NASI)} = 1.1363 \times 10^{-4}$, with a high kurtosis of $\mathbb{Q}^{(NASI)} = 29.31$. In turn, its skewness is $\mathbb{S}^{(NASI)} = -0.18423$, which depicts negative skewness, the maximum and minimum values of the log returns are found to be $X^{(max)}_{(NASI)} = 0.12149$ and $X^{(min)} = -0.1763$ respectively. The value obtained for $\mathbb{S}^{(NASI)} = -0.18423$, which falls between - 0.5 and 0.5, shows that the data is highly skewed and for $\mathbb{Q}^{(NASI)} = 29.31$, it is observed to be too peaked.

4.4.2 The UK stock indices data realisation and description

The data of the UK Stock Market Indices (UKSMI) was obtained from the ForexTime (FXTM) Global trading platform via https//:www.forextime.com. The data set comprised of a sample size of 2076 daily closing price observations, restricted to only trading days (excluding public holidays and weekends) from April 23, 2012 to July 6, 2020. Graphical descriptions of the UK stock market data and its log-returns were presented in Figures 4.3 and 4.4 below.

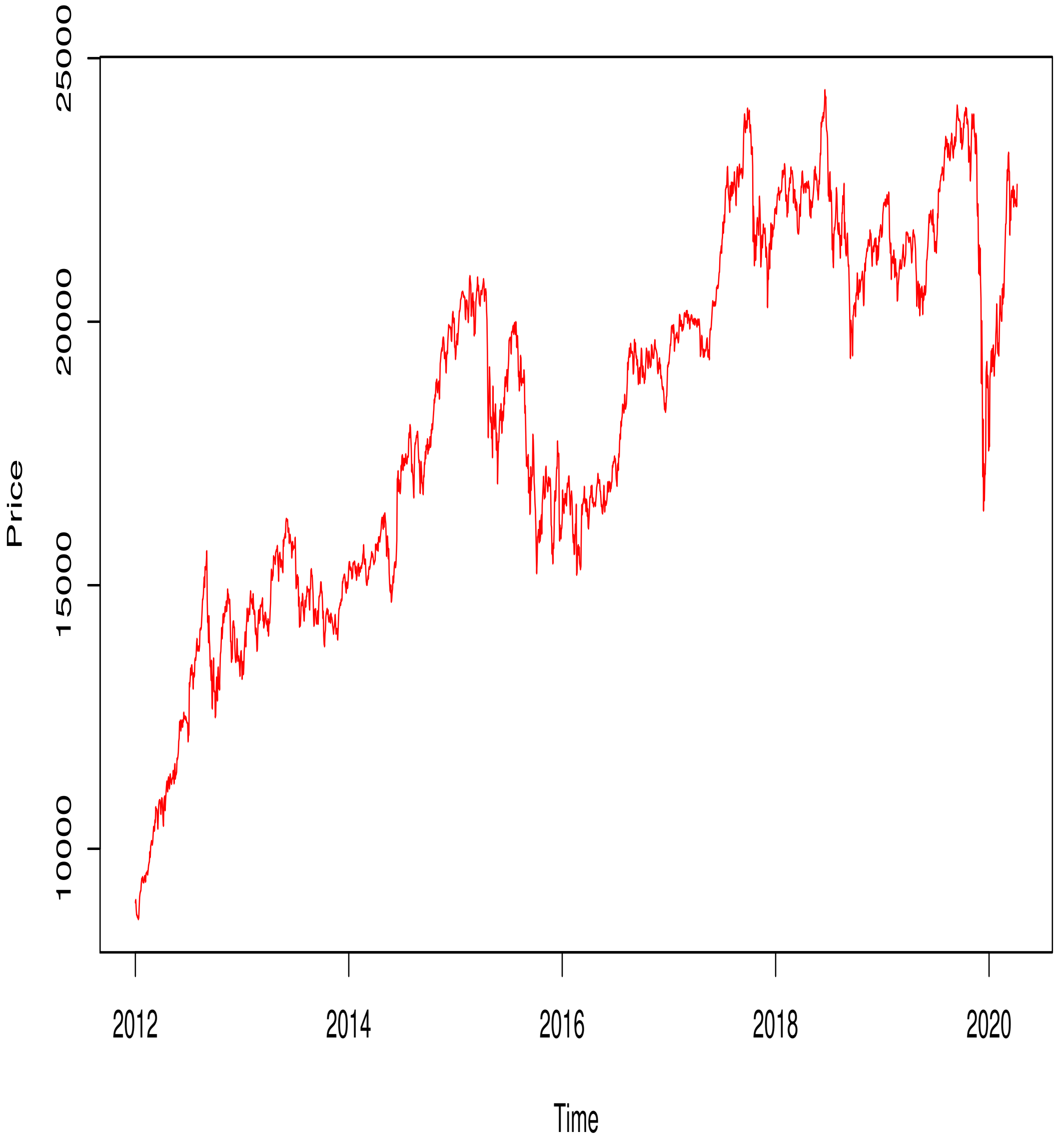

Figure 4.3: Price process $(\tilde{S}_t^{uk})$ of the UK Stock Market Indices from April 23, 2012 to July 6, 2020

The plot of the UKSMI price process ($\tilde{S}_t^{uk}$) was presented in Figure 4.3. A drastic increase in $\tilde{S}_t^{uk}$ from the year 2012 to the year 2015 was observed. More so, sudden and spontaneous changes were observed in the paths of $\tilde{S}_t^{uk}$.

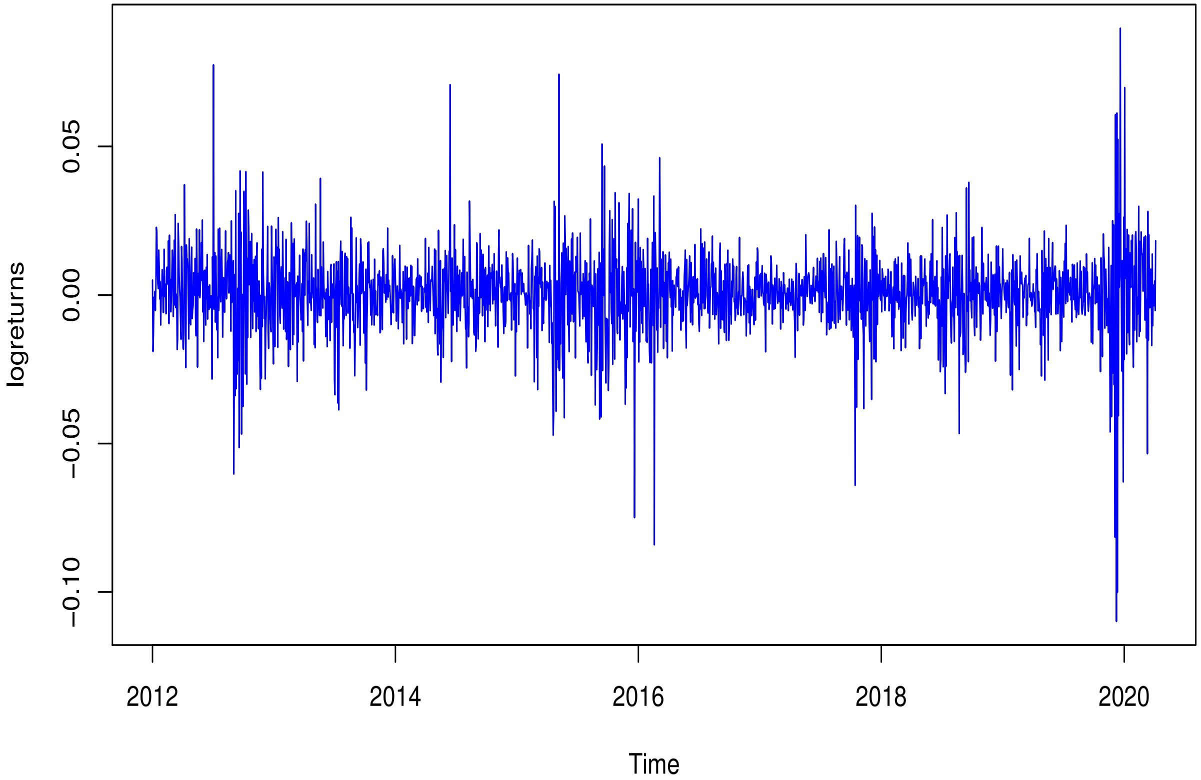

Figure 4.4: Log-returns ($ln\frac{S^{UK}_{t+\Delta t}}{S^{uk}_{t}}$) of the UK stock market indices

The plot of the UKSMI stock price log returns ($ln\frac{S^{uk}_{t+\Delta t}}{\tilde{S}_t}$) was presented in Figure 4.4. The descriptive statistics of the UKSMI log returns were given as follows: $\mathbb{E}^{(UKSMI)} = 4.8 \times 10^{-5}$, $\mathbb{V}^{(UKSMI)} = 1.0629 \times 10^{-2}$ $\mathbb{Q}^{(UKSMI)} = 23.35$, $\mathbb{S}^{(UKSMI)} = -1.3436$, $X^{(max)}_{(UKSMI)} = 0.088898$ and $X^{(min)}_{(UKSMI)} = -0.12193$. Owing to the values obtained for the above descriptions, it can be clearly seen that the paths of $ln\frac{S^{uk}_{t+\Delta t}}{\tilde{S}_t}$ cannot fit into a normal distribution. Moreso, in Figure 4.4, there are evidences of spikes in different sizes.

4.4.3 The Japan stock indices data realisation and description

Similarly, the Japan Stock Market Indices (JSMI) data was also obtained from the same webpage with the UKSMI, as stated in the previous section. The JSMI data set comprise of a sample size of 2075 daily closing price observations from July 5, 2012 to July 6, 2020. Graphical representations of the Japan stock market indices data and its log-returns were presented in Figures 4.5 and 4.6 below.

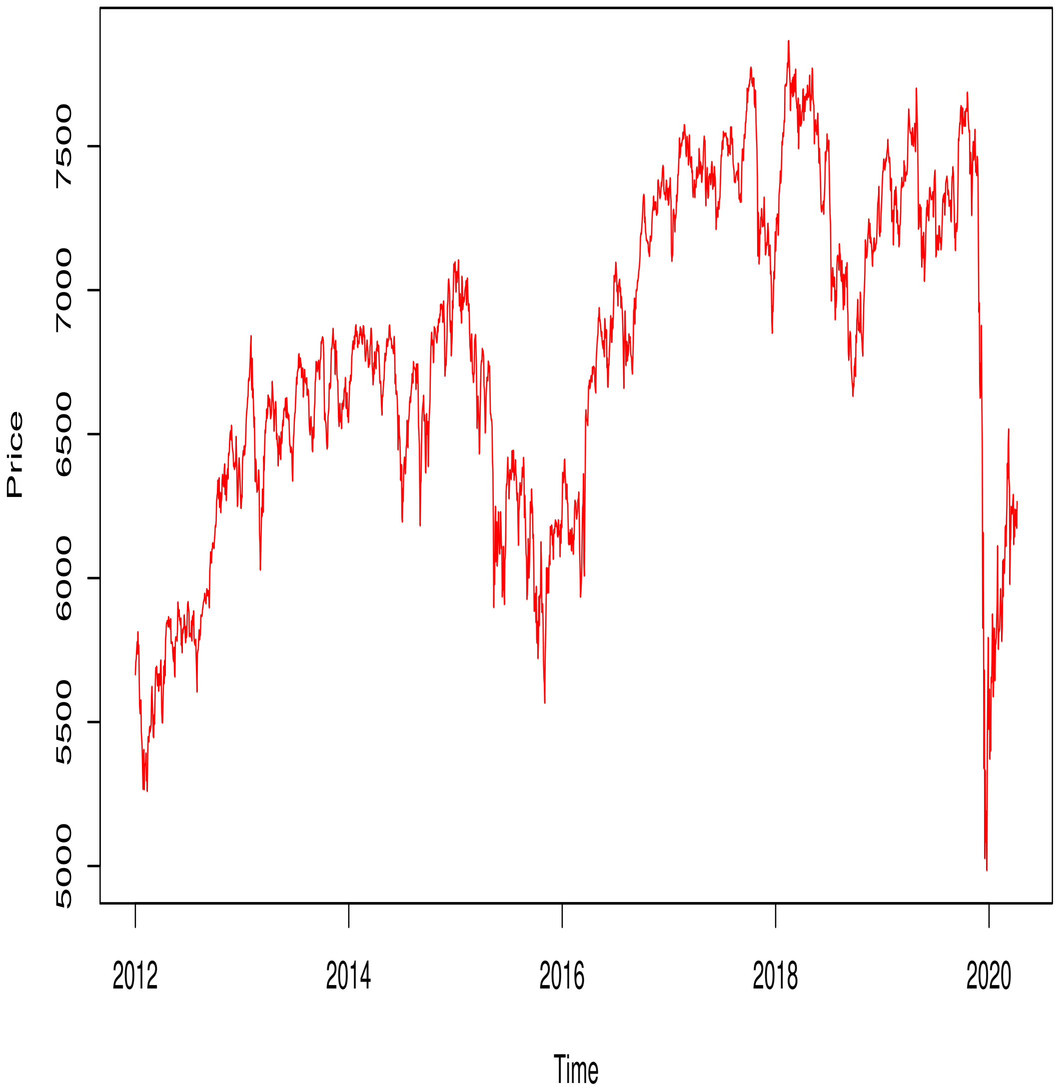

Figure 4.5: Price process ($\tilde{S}_t^{jp}$) of the Japan stock market indices from July 5, 2012 to July 6, 2020

The plot of the price process ($\tilde{S}_t^{jp}$) of the JSMI was presented in Figure 4.5 above. $\tilde{S}_t^{jp}$ was observed to be very low at the early stage of the year 2021 with some high values in the year 2018. Similar to the observations made on $\tilde{S}_t^{uk}$, sudden and spontaneous changes are also observed in the paths of $\tilde{S}_t^{jp}$, more in the latter dates.

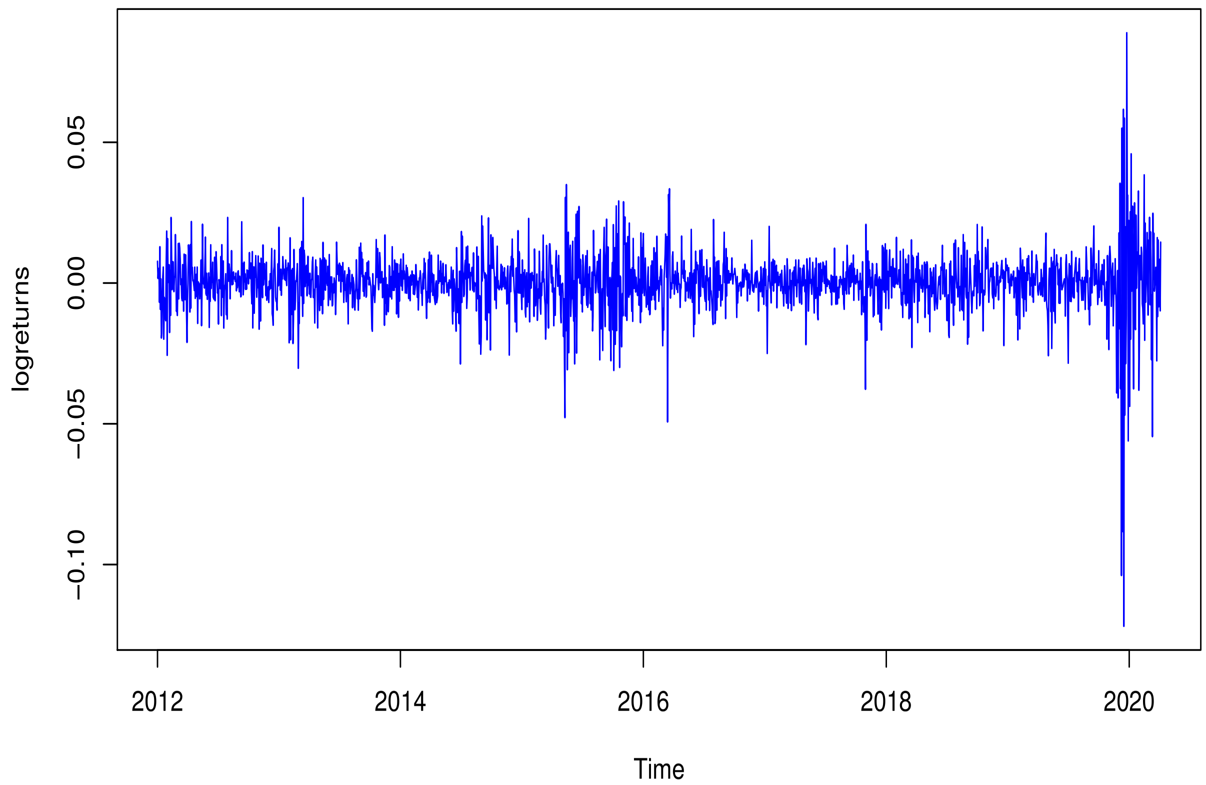

Figure 4.6: Log returns ($ln\frac{S^{jp}_{t+\Delta t}}{S^{jp}_t}$) of the Japan stock market indices

The plot of the JSMI log returns is given in Figure 4.6 and the descriptive statistics of the JSMI log returns were given as follows: $\mathbb{E}^{(UKSMI)} = 4.4 \times 10^{-4}$, $\mathbb{V}^{(JSMI)} = 1.3814 \times 10^{-2}$, $\mathbb{Q}^{(JSMI)} = 11.233$, $\mathbb{S}^{(JSMI)} = -0.5635$, $X^{(max)}_{(JSMI)} = 0.089785$ and $X^{(min)}_{(JSMI)} = -0.10987$. In Figure 4.6, there are more smaller spikes than larger ones, which is not the case in the Nigerian and UK stock markets. However, larger spikes were observed in the downward trends of the distribution of the log returns.

4.5 STUDY FOUR

Detecting jumps in the stock market indices data

In this study, the jump test models derived in section 4.2 to detect jumps in the Nigerian, UK, and, Japan stock market indices data were applied.

4.5.1 Jump test in the NASI data via the particular RMPV models

The jump test for the empirical data of Nigerian All Share Index via the models given in equations (4.47)-(4.55) shall be conducted using a 5%(0.05) level of significance. In view of the above, the hypotheses were set as:

$$H_0 : X_{(\Delta,t)}^{(NASI)} \in Smsv^c$$

.

$$H_1 : X_{(\Delta,t)}^{(NASI)} \in Smsv^j$$

In the analysis, the Rcodes were used to compute asymptotic variance φ_{RMPV} for the particular cases, that is $\varphi_{RBV}, \varphi_{RTV}, \varphi_{RQV}, \varphi_{RPV}, \varphi_{RH_XV}, \varphi_{RH_pV}$, φ_{ROV}, φ_{RNV}, and φ_{RDV} were computed. The different values of $\hat{\mathrm{Z}}_{\mathrm{m}}$ for each of the value of $m = 2, \ldots, 10$ as given in equations (4.47)-(4.55) as well as their respective p-values (estimate probability value which was used to reject or not the null hypothesis) were also computed. Moreso, the realised variance $\{X\}_{\Delta,t}^{(2)}$ of the NASI data and the respective r^{th} power variation process were calculated. The test criterion: Reject H_0, If p-value < 0.05. The following results were obtained from the jump test analysis using the NASI data in the particular RMPV models and were presented in Table 4.1.

Table 4.1: Results of the jump test in the NASI data via the RMPV model

$\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}$	$\varphi_{RMPV}(m,\mu_r)$	$\hat{\mathrm{Z}}_{\mathrm{m}}$	p-value	$\{X\}_{\Delta,t}^{(2)}$	r^{th} RPV
m=2	0.6090	3.8995	9.64 e-05	0.7251	0.6715
m=3	1.0613	8.0285	9.87 e-15	0.7251	0.5794
m=4	1.3770	8.7791	1.65 e-18	0.7251	0.5436
m=5	1.6053	9.3854	6.26 e-21	0.7251	0.5156
m=6	1.7769	9.8718	5.52 e-23	0.7251	0.4932
m=7	1.9100	10.2100	1.79 e-24	0.7251	0.4765
m=8	2.0161	10.6183	2.45 e-26	0.7251	0.4594
m=9	2.1026	10.9508	6.59 e-28	0.7251	0.4453
m=10	2.1744	11.2778	1.69 e-29	0.7251	0.4321

Table 4.1 reports the computed results for the particular cases of the jump test model in the NASI data. The components of the model include $\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}$, φ_{RMPV}, $\hat{\mathrm{Z}}_{\mathrm{m}}$, p-value, $\{X\}_{\Delta,t}^{(2)}$, and the RMPV r^{th} power variation. The particular values of the asymptotic variances φ_{RMPV} are obtained using rcodes in terms of μ_r given in study one. The values of $\hat{\mathrm{Z}}_{\mathrm{m}}$, which range between 3.8995 and 11.2778, represent the test statistics on a 5%(0.05) level of significance under the null hypothesis (H_0) of no jump in the NASI log returns. The p-values range between 9.64 e-05 and 1.69e-29, which is the probability of rejecting H_0 or not. $\{X\}_{\Delta,t}^{(2)}$ was obtained as a constant in all the particular cases,which is the realised variance needed in computation of $\hat{\mathrm{Z}}_{\mathrm{m}}$. The null hypothesis was rejected based on the shreds of evidence in Table 4.1 and concerning the p-values obtained, which falls far below 0.05. This means that there are jumps in the NASI log returns.

4.5.2 Jump test in the UKSMI data via the particular RMPV models

Similarly, the presence of jumps in the empirical data of the UK Stock Indices via the models given in subsection 4.3.1 was tested.
The hypotheses were given as:

$$H_0 : X^{(UKSMI)}_{(\Delta,t)} \in Smsv^c$$
$$H_1 : X^{(UKSMI)}_{(\Delta,t)} \in Smsv^j$$

The jump test on a significant level, which was chosen as 5%(0.05), was carried out in this subsection. In the analysis, the Rcodes were used to compute asymptotic variance φ_{RMPV} for the particular cases, given as: $\varphi_{RBV}, \varphi_{RTV}, \varphi_{RQV}, \varphi_{RPV}, \varphi_{RH_XV}, \varphi_{RH_pV}, \varphi_{ROV}, \varphi_{RNV}$, and φ_{RDV}. The different values of $\hat{\mathrm{Z}}_{\mathrm{m}}$, were computed for each of the value of $m = 2, \ldots, 10$ as given in equations (4.47)-(4.54) as well as their respective p-values (estimated probability value which was used to reject the null hypothesis or not to reject H_0). Also, the realised variance $\{X\}^{(2)}_{\Delta,t}$ of the UKSMI data and the respective r^{th} power variation process were calculated. The test criterion: Reject H_0, If p-value < 0.05. The following results were obtained from the jump test analysis using the UKSMI data in the particular RMPV models and presented in Table 4.2.

Table 4.2: Results of the jump test in the UKSMI data via the RMPV model

$\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}$	$\varphi_{RMPV}(m,\mu_r)$	$\hat{\mathrm{Z}}_{\mathrm{m}}$	p-value	$\{X\}_{\Delta,t}^{(2)}$	r^{th} RPV
m=2	0.6090	2.4553	$5.64e-04$	0.5354	0.4715
m=3	1.0613	5.0285	$9.87e-11$	0.5354	0.3794
m=4	1.3770	5.7791	1.65 e-12	0.5354	0.3854
m=5	1.6053	6.3854	6.26 e-15	0.5354	0.3632
m=6	1.7769	6.8718	5.52 e-18	0.5354	0.3214
m=7	1.9100	8.2100	1.79 e-21	0.5354	0.3154
m=8	2.0161	8.6183	2.45 e-24	0.5354	0.2954
m=9	2.1026	8.9508	6.59 e-25	0.5354	0.2623
m=10	2.1744	9.2778	1.69 e -27	0.5354	0.2412

The Table 4.2 shows the computed results of the jump test particular cases in the UKSMI log-return, which include: $\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}, \varphi_{RMPV}, \hat{\mathrm{Z}}_{\mathrm{m}}$, p-value, $\{X\}_{\Delta,t}^{(2)}$, and the RMPV r^{th} power variation. The values of $\hat{\mathrm{Z}}_{\mathrm{m}}$ in the UKSMI data ranged between 2.4553 and 9.2778, the p-values ranged between 5.64e-04 and 1.69e-27, $\{X\}_{\Delta,t}^{(2)}$ was found to be 0.5354 in all the particular cases. The null hypothesis was rejected in all the particular cases based on the shreds of evidence in Table 4.2 and concerning the p-values obtained, which fell below 0.05. This means that there are jumps in the UKSMI log returns.

4.5.3 Jump test in the JSMI data via the Particular RMPV models

Here, the presence of jumps in the empirical data of the Japan Stock Indices via the models given in subsection 4.3.1 was tested.

The hypotheses were given as

$$H_0 : X_{(\Delta,t)}^{(JSMI)} \in Smsv^c$$
$$H_1 : X_{(\Delta,t)}^{(JSMI)} \in Smsv^j$$

The jump test on a significant level, which was chosen as 5%(0.05), was carried out in this subsection. In the analysis, the Rcodes were used to compute asymptotic variance φ_{RMPV} for the particular cases, given as: $\varphi_{RBV}, \varphi_{RTV}, \varphi_{RQV}, \varphi_{RPV}$, $\varphi_{RH_XV}, \varphi_{RH_pV}, \varphi_{ROV}, \varphi_{RNV}$, and φ_{RDV}. The different values of $\hat{\mathrm{Z}}_{\mathrm{m}}$, were computed for each of the value of $m = 2, \ldots, 10$ as given in equations (4.47)-(4.54) as well as their respective p-values (estimated probability value which was used to reject H_0 or not). Also, the realised variance $\{X\}_{\Delta,t}^{(2)}$ of the JSMI data and the respective r^{th} power variation process were calculated. The test criterion: Reject H_0, If p-value < 0.05. The following results were obtained from the jump test analysis using the JSMI data in the particular RMPV models and presented in Table 4.3.

Table 4.3: Results of the jump test in the JSMI data via the RMPV model

$\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}$	$\varphi_{RMPV}(m,\mu_r)$	$\hat{\mathrm{Z}}_{\mathrm{m}}$	p-value	$\{X\}_{\Delta,t}^{(2)}$	r^{th} RPV
m=2	0.6090	3.4553	$5.64e-03$	0.4554	0.4715
m=3	1.0613	6.0285	$9.87e-11$	0.4554	0.3794
m=4	1.3770	6.7791	1.65 e-12	0.4554	0.3854
m=5	1.6053	7.3854	6.26 e-15	0.4554	0.3632
m=6	1.7769	8.8718	5.52 e -18	0.4554	0.3214
m=7	1.9100	9.2100	1.79 e -21	0.4554	0.3154
m=8	2.0161	9.6183	2.45 e -24	0.4554	0.2954
m=9	2.1026	10.9508	6.59 e -25	0.4554	0.2623
m=10	2.1744	11.3178	1.69 e -27	0.4554	0.2412

Table 4.3 above shows the computed results for the jump test particular cases, which include: $\{X\}_{\Delta,t}^{(r_1,\ldots,r_m)}$, φ_{RMPV}, $\hat{\mathrm{Z}}_{\mathrm{m}}$, p-value, $\{X\}_{\Delta,t}^{(2)}$, and the RMPV r^{th} power variation in the JSMI log returns. The values of $\hat{\mathrm{Z}}_{\mathrm{m}}$ in the JSMI data range between 3.4553 and 11.3178, the p-values range between 5.64e-03 and 1.69e-27, $\{X\}_{\Delta,t}^{(2)}$ was found to be 0.4554 in all the particular cases. Based on the shreds of evidence in Table 4.3, concerning the p-values obtained, which fall below 0.05, H_0 was rejected in all cases. This means that there are jumps in the JSMI log returns.

4.5.4 Plots showing visible jumps in the sampled paths of the stock market indices log returns

The empirical evidence of the presence of jumps via the RMPV jump test models in the stock market log returns has been established in subsections 4.4.1-4.4.3. The evidence of jumps in $\Delta(lnS_t)$ were buttressed further by observing the jumps in the sampled paths of the three stock market indices log returns (see Figures 4.7, 4.8 and 4.9 below). These observations were necessary to visualise the upward and downward jumps, in different sizes and enabled us build suitable dynamics for the log returns.

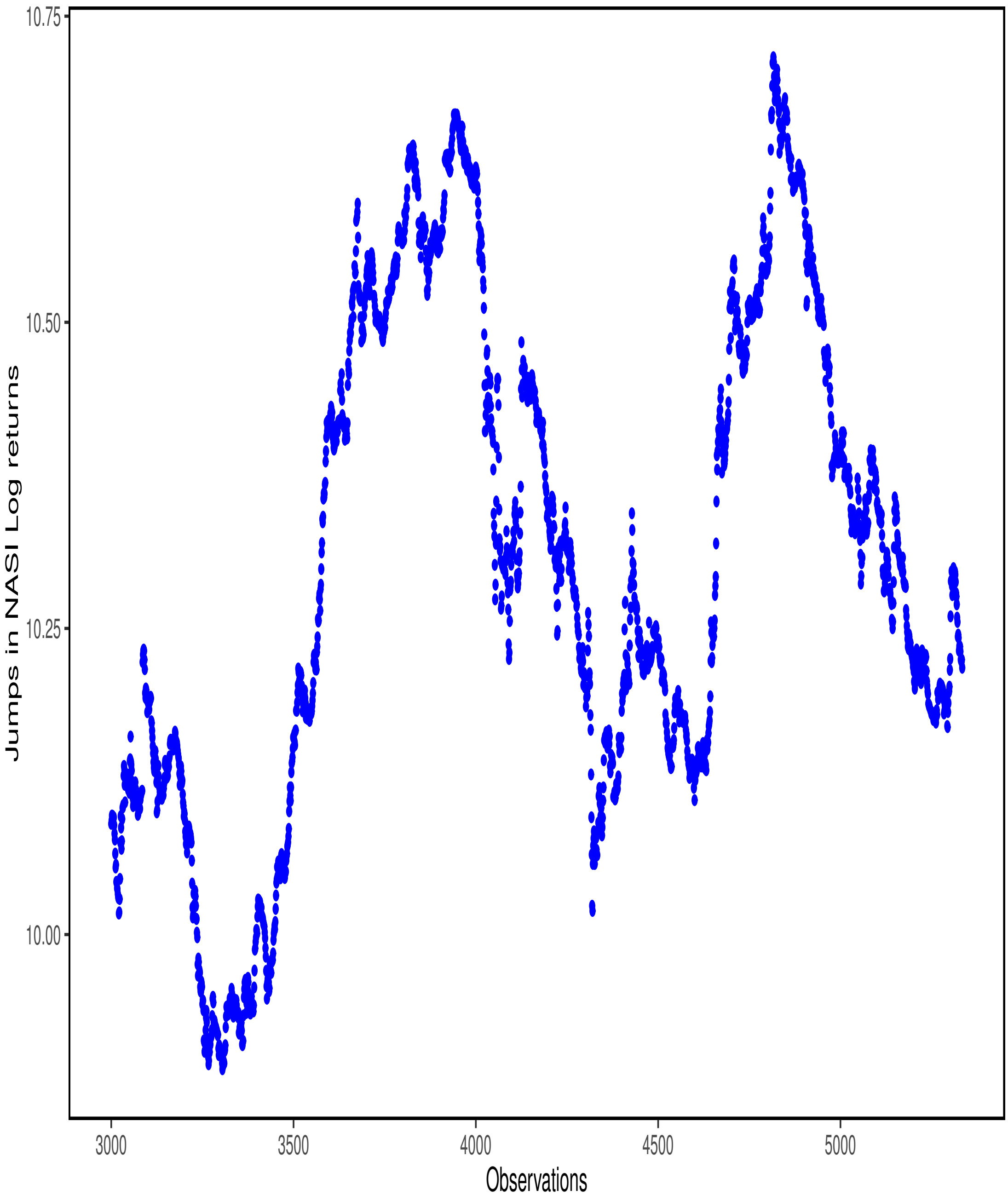

Figure 4.7: The jumps in the sampled NASI log returns

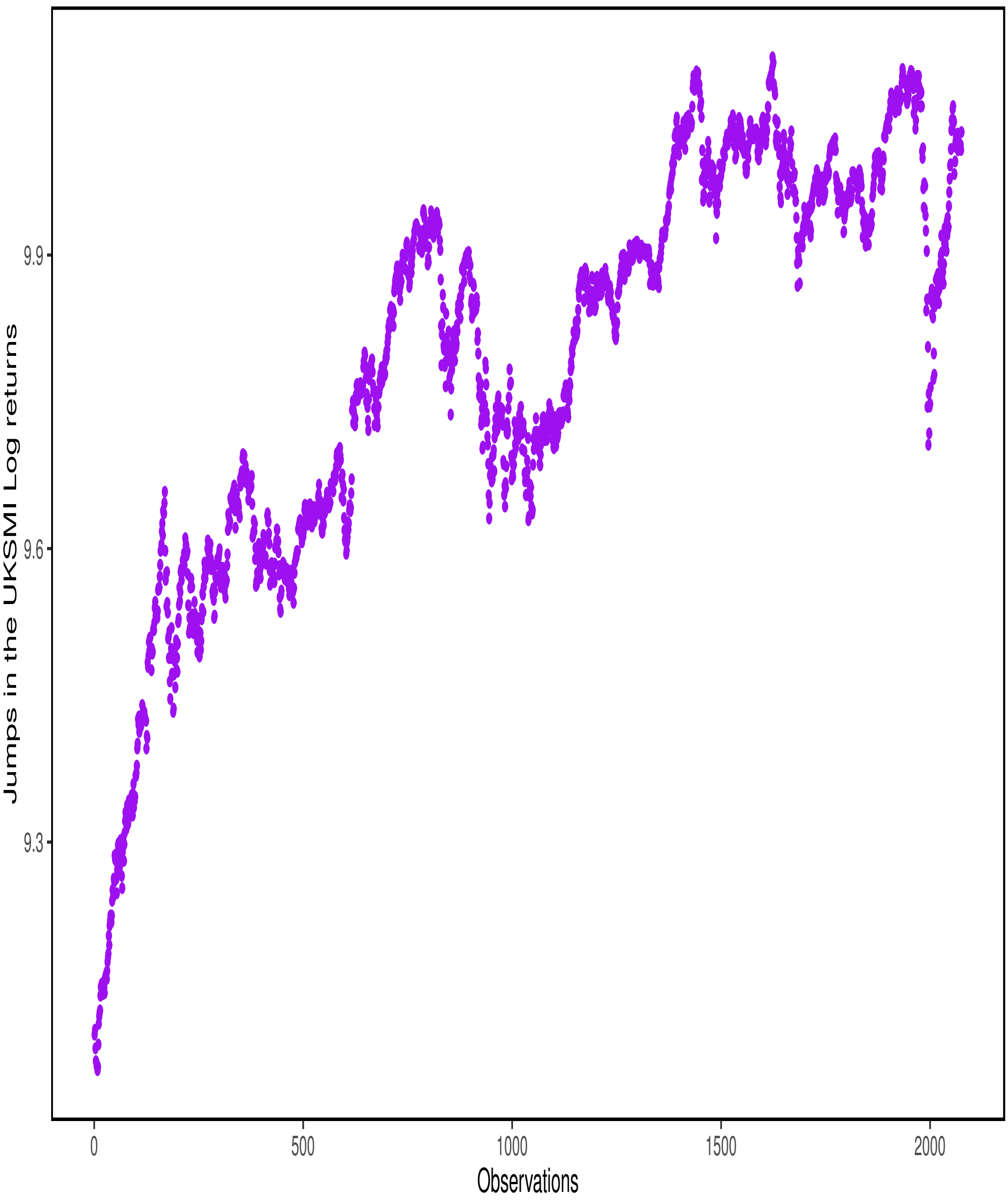

Figure 4.8: The jumps in the sampled UKSMI log returns

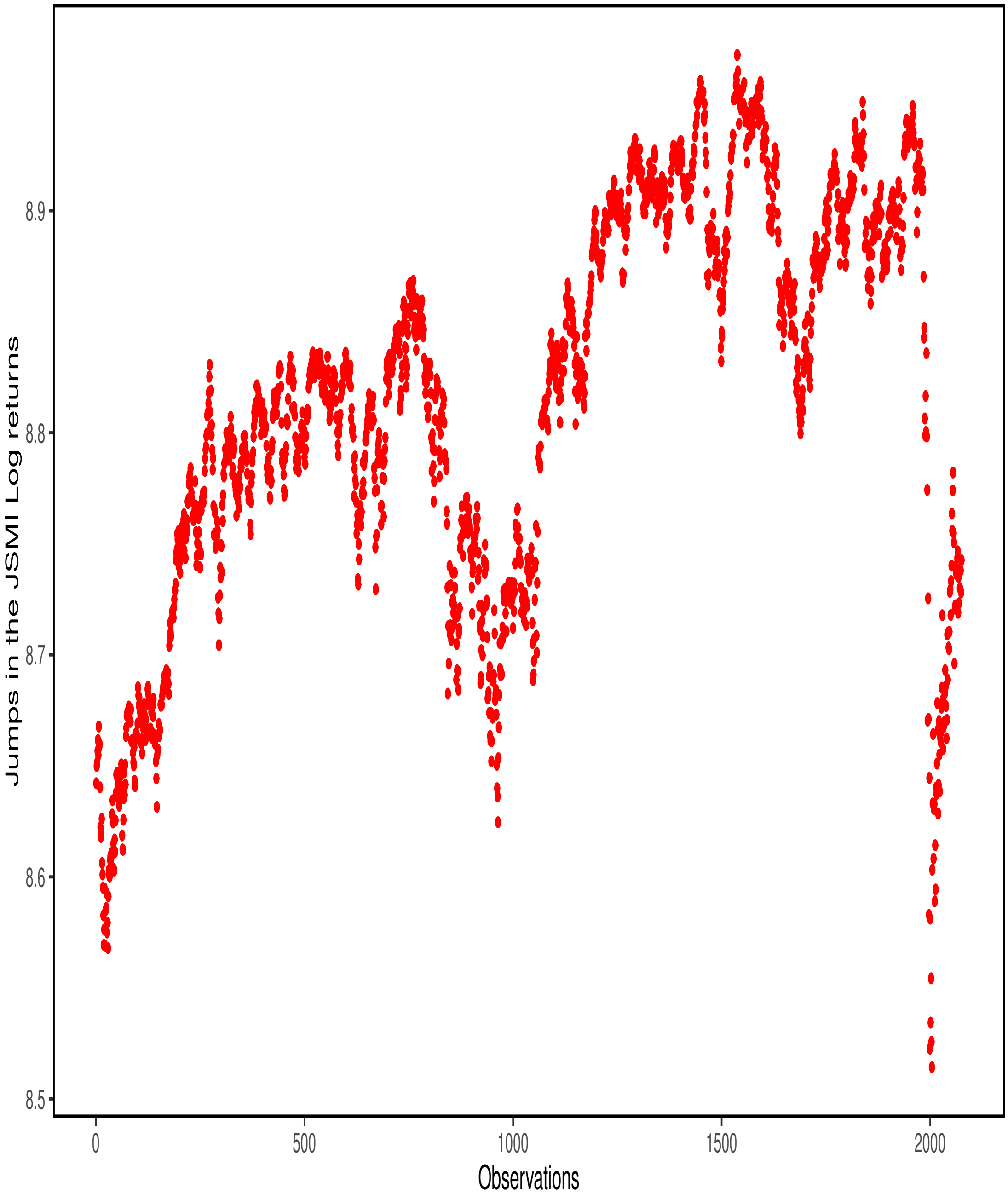

Figure 4.9: The jumps in the sampled JSMI log returns

The sample paths showing the jumps or discontinuous paths were presented in Figures 4.7, 4.8 and 4.9. In order to observe these jumps vividly, a smaller sample comprising 2334 observations was selected in the case of the Nigerian stock market, as shown in Figure 4.7. The jumps in the price process in the three stock markets were quite visible as shown in Figures 4.7-4.8. Also, upward and downward jumps of different jump sizes were observed in Figures 4.7-4.9, showing that arrival times, jump size and intensities are independent. However, more larger jumps were observed between 1500 and 2000 observations in the Japan market as can be seen in Figure 4.9.

4.6 STUDY FIVE

Generalised asymmetric jump-diffusion processes for stock price modelling

In this study, a family of skewed jump-diffusion models with non-zero location parameters and scale parameters for upward and downward distributions of the random jump processes given the dynamics of the log returns of stock price process was considered.

4.6.1 An asymmetric jump-diffusion model driven by the AL distribution

An extension of a stock price model which is driven by diffusion was considered here. Let $\tilde{S}_t$ be the price of a stock, which satisfies the Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, F)$ such that the dynamics of $\tilde{S}_t$ is given as:

$$d\tilde{S}_t = \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t + \tilde{S}_t J(Q_j) dN_t \tag{4.56}$$

where, μ is the mean return rate of the diffusive process, and σ is the diffusive volatility, W_t is a standard Brownian motion, N_t is a Poisson process with respect to the filtration F having a constant jump rate (jump intensity) λ; and $J(Q_j)$ is a non-constant jump amplitude (random jump process), where also, the random variables W_t, N_t and $J(Q_j)$ are independent. The random jump process above can be expressed as:

$$\int_{\zeta_t}^{\zeta_{t+\Delta_t}} J(Q_j) dN_t = \sum_{j=1}^{\Delta N_t} J(Q_j), \qquad \zeta_{t+\Delta_t} - \zeta_t = \Delta N_t \tag{4.57}$$

given that the Q_j's are i.i.d random variables, the probability density function of N_t is

$$\mathbb{P}(N_t = k) = \frac{(\lambda \Delta t)^k exp(-\lambda \Delta t)}{k!} \tag{4.58}$$

The solution of equation (4.56) can be obtained by the $It\hat{o}'s$ formula for jump diffusion and is obtained as:

$$\Delta(\ln \tilde{S}_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_t + J(Q_j)\Delta N_t \tag{4.59}$$

In the work of Kou (2002), the jump intensity λ, was assumed as the jump intensity for both the upward and downward jump processes. However, based on the results obtained from recent empirical studies in Lau *et al.* (2019), it was found that the jump intensities of both processes are independent. Therefore, in this research, skewed jump-diffusion models with independent jump intensities λ_j^u and λ_j^d; with jump frequencies $n(Q_t^u)$ and $n(Q_j^d)$ for Q_j^u and Q_j^d were proposed.
So,

$$J(Q_j)\Delta N_t = J(Q_j^u)\Delta N_t^u + J(Q_j^d)\Delta N_t^d \tag{4.60}$$

Then, equation (4.59) can be expressed as:

$$\Delta(\ln \tilde{S}_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\Delta W_t + J(Q_j^u)\Delta N_t^u + J(Q_j^d)\Delta N_t^d \tag{4.61}$$

4.6.1.1 Asymmetric Laplace distribution and its properties

The asymmetric Laplace distribution henceforth named as $AL^*(\mu, \sigma^2, \kappa)$ distribution according to Kozubowski and Podgorski (2000) and Kotz *et al* (2001); is a three parameter skewed Laplace distribution with a location and scale parameter which are respectively μ and σ, its skewness is indexed by the parameter $\kappa > 0$.

Definition 4.7: Probability density function of the $AL^*(\mu, \sigma, \kappa)$ distribution
The $AL^*(\mu, \sigma^2, \kappa)$ distribution has a probability density function (pdf) given as:

$$f(x; \mu, \sigma, \kappa) = \frac{\sqrt{2}}{\sigma}\frac{\kappa}{(1+\kappa^2)}\begin{cases} exp\left(-\frac{\sqrt{2}\kappa}{\sigma}(x-\mu)\right), & x \geq \mu \\ \\ exp\left(-\frac{\sqrt{2}}{\sigma\kappa}(x-\mu)\right), & x < \mu \end{cases} \tag{4.62}$$

where, $\sigma > 0, -\infty\mu < \infty(\mu \in \Re)$.
The cumulative density function (cdf) is

$$F(x;\mu,\sigma,\kappa)=\frac{\sqrt{2}}{\sigma}\begin{cases}1-\frac{\kappa}{(1+\kappa^2)}exp\Bigg(-\frac{\sqrt{2}\kappa}{\sigma}(x-\mu)\Bigg), & x\geq\mu\\ \\ \frac{\kappa}{(1+\kappa^2)}exp\Bigg(\frac{\sqrt{2}}{\sigma\kappa}(x-\mu)\Bigg), & x<\mu\end{cases}\tag{4.63}$$

The $AL^*(\mu,\sigma^2,\kappa)$ distribution is a flexible and robust kind of distribution and has some very interesting properties which can be useful in financial modelling.

Remark 4.1

The probability density function of the $AL^*(\mu,\sigma,\kappa)$ distribution given in equation (4.62) can be expressed as a mixture of two exponential densities (see Kotz *et al.*, 2001, pg. 172) given below:

$$f(x)=p_\kappa\alpha_1 exp(-\alpha_1(x-\mu))1_{[\mu,\infty)}(x)+q_\kappa\alpha_\kappa exp(\alpha_2(x-\mu))1_{(-\infty,\mu)}(x)\tag{4.64}$$

where, $p_\kappa=\frac{1}{1+\kappa^2}$, $q_\kappa=\frac{k}{(1+\kappa^2)}$, $\alpha_1=\frac{\sqrt{2}\kappa}{\sigma}$, $\alpha_2=\frac{\sqrt{2}}{\sigma\kappa}$, $p_\kappa,q_\kappa\geq 0$, $p_\kappa+q_\kappa=1$
Note also that $\mathbb{P}(x<\mu)=1-F(\mu)=\frac{1}{1+\kappa^2}=p_\kappa$, which implies that κ controls the tail probabilities.

4.6.2 The density function of the asymmetric Laplace jump-diffusion process

Considering the processes in equations (4.56) and (4.59), their densities are majorly determined by the distribution of the jump process Q_j, which gives the different types of jump-diffusion models in literature. However, the distributions of $J(Q_j^u)$ and $J(Q_j^d)$ was assumed as an Asymmetric Laplace distributions, such that, $Q_j^u\sim AL^*(\mu_j,(\sigma_j^u)^2,\kappa)$ and $Q_j^d\sim AL^*(\mu_j,(\sigma_j^d)^2,\kappa)$, where, μ is the mean of the jump process, σ is the volatility of the jump process and κ controls the skewness of the distribution of the jump sizes, were viewed here. In the next subsection, the density of the process subject to $Q_j\sim AL^*(\mu,\sigma^2,\kappa)$ was obtained. In the next theorem, the density of the jump diffusion process of the log returns $\Delta(\ln\tilde{S}_t)$ when $Q_j\sim AL^*(\mu,\sigma^2,\kappa)$ was determined.

Theorem 4.1

The probability density of the $AL^*(\mu, \sigma^2, \kappa)$ jump diffusion process is given by

$$
\begin{aligned}
f_{\Delta(\ln \tilde{S}_t)}(x) =& \frac{(1-\lambda\Delta t)}{\sigma\sqrt{\Delta t}}\varphi\Big(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\Big) + \Delta t\Big(p_\kappa \alpha_2 \lambda_j^u exp\Big(\frac{2\alpha_1\mu_j + \alpha_1\sigma^2\Delta t}{2}\Big) \\
& exp\Big(-\Big(x-\Big(\mu-\frac{1}{2}\sigma^2\Big)\Delta t\Big)\alpha_1\Big)\Phi_a\big(\mu_j\big) + q_k\alpha_2\lambda_j^d exp\Big(\frac{2\alpha_2\mu_j + \alpha_2\sigma^2\Delta t}{2}\Big) \\
& exp\Big(-\Big(x-\Big(\mu-\frac{1}{2}\sigma^2\Big)\Delta t\Big)\alpha_2\Big)\Phi_b\big(-\mu_j\big)\Big)
\end{aligned}
\tag{4.65}
$$

where, $\Phi_a(\mu_j)$ and $1-\Phi_b(\mu_j) = \Phi_b(-\mu_j)$ are the cumulative normal distributions of $x \sim N(x; (-\mu_d\Delta t) - \alpha_1\sigma^2\Delta t, 2\sigma^2\Delta t)$ and $x \sim N(x; (-\mu_d\Delta t) + \alpha_2\sigma^2\Delta t, 2\sigma^2\Delta t)$ respectively at $x = \mu_j$ and $x = -\mu_j$.

Proof.:

Let $\tilde{S}_t$ be the stock price process defined on $(\Omega, F, \mathbb{P}, \mathbb{F})$ such that the dynamics of $\tilde{S}_t$ is given by:

$$d\tilde{S}_t = \mu\tilde{S}_t dt + \sigma\tilde{S}_t dW_t + \tilde{S}_t J(Q_j)dN_t \tag{4.66}$$

where, $S_0 > 0, \mu, \sigma, W_t$ are as previously defined. The random jump process in equation (4.66) above is given as:

$$\int_{\zeta_t}^{\zeta_{t+\Delta t}} J(Q_j)dN_t = \sum_{i=1}^{\Delta N_t} J(Q_j), \tag{4.67}$$

where, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity rate λ and density

$$\mathbb{P}(N_t = k) = \frac{(\lambda\Delta t)^k exp(\lambda\Delta t)}{k!} \tag{4.68}$$

By the $It\hat{o}'s$ lemma, the dynamics in equation (4.66) can be obtained as equation (4.59).

Thus, equation (4.59) implies $\tilde{S}_t = S_0 exp(\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_t + \sum_{i=1}^{\Delta N_t} Q_j$

Also, equation (4.59) is approximately:

$$\Delta(\ln \tilde{S}_t) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t W_t} + VQ_j \tag{4.69}$$

where, V is a Bernoulli random variable such that $\mathbb{P}(V = 1) = \lambda\Delta t$ and $\mathbb{P}(V = 0) = 1 - \lambda\Delta t$

Hence, the random jump process Q_j above satisfies:

$$\sum_{j=1}^{\Delta N_t} Q_j \stackrel{d}{=} \begin{cases} Q, & \mathbb{P} = \lambda\Delta t. \\ \\ 0, & \mathbb{P} = 1 - \lambda\Delta t \end{cases} \tag{4.70}$$

Thus, the density of the process in equation (4.69), can be expressed as:

$$f_{\Delta(\ln \tilde{S}_t)}(x) = (1 - \lambda\Delta t)f_{X_t}(x) + \lambda\Delta t f_{X_t+Q_t}(x) \tag{4.71}$$

From the above, given that $X_t = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_t$ and $Q_t = \sum_{j=1}^{\Delta N_t} Q_j = \sum_{i=1}^{\Delta N_t^u} Q_j^u + \sum_{i=1}^{\Delta N_t^d} Q_j^d$, such that

$$f_{Q_t^u}(x) = p_k\alpha_1\lambda_j^u exp(-\alpha_1(x - \mu))\mathbf{1}_{[\mu,\infty)}(x) \tag{4.72}$$

and

$$f_{Q_t^d}(x) = q_k\alpha_2\lambda_j^d \exp(\alpha_2(x - \mu))\mathbf{1}_{(-\infty,\mu)}(x) \tag{4.73}$$

Then,

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\bigg(- \frac{(x - (\mu - \frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\bigg) = \varphi\bigg(\frac{x - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\bigg) \tag{4.74}$$

and

$$f_{Q_t}(x) = p_k\alpha_1\lambda_j^u exp(-\alpha_1(x - \mu))\mathbf{1}_{[\mu,\infty)}(x) + q_k\alpha_2\lambda_j^d \exp(\alpha_2(x - \mu))\mathbf{1}_{(-\infty,\mu)}(x) \tag{4.75}$$

In equation (4.74), the density $f_{X_t}(x)$ is known already, and hence, $f_{X_t+Q_t}(x)$ by the convolution of densities was obtained. It followed for $z \in \mathbb{R}$ that:

$$f_{X_t+Q_t}(x) = \int_{-\infty}^{\infty} f_X(z-x) f_Q(x) dx \tag{4.76}$$

It followed from the above that:

,

$$\begin{aligned} f_{X_t+Q_t}(x) = \int_{-\infty}^{\infty} & \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\Big(- \frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\Big) \\ & \Big(p_k\alpha_1\lambda_j^u exp\big(-\alpha_1(x-\mu_j)\big)\Big)\mathbf{1}_{[\mu,\infty)}(x) \\ & + q_k\alpha_2\lambda_j^d exp(\alpha_2(x-\mu_j))\mathbf{1}_{(-\infty,\mu)}(x)\Big)dx \end{aligned} \tag{4.77}$$

$$\begin{aligned} &= \int_{\mu_j}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right)\mathbb{P}_k\alpha_1\lambda_j^u exp\big(-\alpha_1(x-\mu)\big)\right)dx \\ &\quad + \int_{\infty j}^{\mu_j} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) q_k\alpha_2\lambda_j^d exp\big(-\alpha_2(x-\mu_j)\big)\right)dx \end{aligned}$$

$$\begin{aligned} &= \int_{\mu j}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} p_k\alpha_1\lambda_j^u exp\left(-\alpha_1(x-\mu_j) - \frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx \\ &\quad + \int_{-\infty}^{\mu_j} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} q_k\alpha_2\lambda_j^d exp\left(-\alpha_2(x-\mu_j) - \frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx \end{aligned} \tag{4.78}$$

Let the diffusive drift $\mu_d = \mu - \frac{1}{2}\sigma^2$, then, equation (4.78) is expressed as:

$$\begin{aligned} f_{X_t+Q_t}(x) = & \int_{\mu j}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} p_k\alpha_1\lambda_j^u exp\left(\mathcal{M}\right)\right)dx \\ & + \int_{-\infty}^{\mu_j} \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} q_k\alpha_2\lambda_j^d exp\left(\mathcal{N}\right)\right)dx \end{aligned}$$

where,

$$\mathcal{M} = \frac{-2\alpha_1\sigma^2\Delta t(x-\mu_j) - (x^2 - 2x(z-\mu_d\Delta t) + (z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}$$

and

$$\mathcal{N} = \frac{2\alpha_2\sigma^2\Delta t(x-\mu_j) - (x^2 - 2x(z-\mu_d\Delta t) + (z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}$$

Then,

$$\begin{aligned} f_{X_t+Q_t}(x) =& p_k\alpha_1\lambda_j^u exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_1\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) \\ & \cdot\int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left(\frac{-2\alpha_1\sigma^2\Delta tx-(x^2-2x(z-\mu_d\Delta t)}{2\sigma^2\Delta t}\right)\right)dx \\ & + q_k\alpha_2\lambda_j^d exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_2\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) \\ & \cdot\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left(\frac{2\alpha_2\sigma^2\Delta ty-(x^2-2x(z-\mu_d\Delta t))}{2\sigma^2\Delta t}\right)\right)dx \end{aligned} \tag{4.79}$$

$$\begin{aligned} =& p_k\alpha_1\lambda_j^u exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_1\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) \\ & \cdot\int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left(\frac{-x^2+2x((z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)}{2\sigma^2\Delta t}\right)\right)dx \\ & + q_k\alpha_2\lambda_j^d exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_2\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) \\ & \cdot\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\left(\frac{-x^2+2x((z-\mu_d\Delta t)-\alpha_2\sigma^2\Delta t)}{2\sigma^2\Delta t}\right)\right)dx \end{aligned} \tag{4.80}$$

Using $y^2 - 2yb = (y-b)^2 - b^2$ in equation (4.80),

$$\begin{aligned} f_{X_t+Q_t}(x) =& p_k\alpha_1\lambda_j^u exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_1\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right)\int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\right. \\ & \left. exp\left(-\frac{(x-(z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2-((z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx \\ & + q_k\alpha_2\lambda_j^d exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_2\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right)\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\right. \\ & \left. exp\left(-\frac{(x-(z-\mu_d\Delta t)+\sigma^2\Delta t)^2-((z-\mu_d\Delta t)-\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx \end{aligned}$$

$$\begin{aligned} =& p_k\alpha_1\lambda_j^u exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_1\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) exp\left(\frac{((z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right) \\ & \int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(x-(z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx + q_k\alpha_2\lambda_j^d exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) \\ & exp\left(\frac{2\alpha_2\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) exp\left(\frac{((z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\right. \\ & \left. exp\left(-\frac{(y-(z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx \end{aligned}$$

$$
\begin{aligned}
f_{X_t+Q_t}(x) =& p_k\alpha_1\lambda_j^u exp\left(\frac{2\alpha_1\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) exp\left(\frac{-2\alpha_1\sigma^2\Delta t}{2\sigma^2\Delta t}\right) exp\left(\frac{((z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\\
&\int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(x-(z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx + q_k\alpha_2\lambda_j^d\\
&exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right) exp\left(\frac{2\alpha_2\mu_j\sigma^2\Delta t}{2\sigma^2\Delta t}\right) exp\left(\frac{((z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\\
&\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(x-(z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx
\end{aligned}
$$

$$
\begin{aligned}
=& p_k\alpha_1\lambda_j^u exp(\alpha_1\mu_j) exp(-(z-\mu_d\Delta t)\alpha_1) exp\left(\frac{\alpha_1\sigma^2\Delta t}{2}\right)\\
&\int_{\mu j}^{\infty}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(x-(z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx\\
&+ q_k\alpha_2\lambda_j^d exp(\alpha_2\mu_j) exp((x-\mu_d\Delta t)\alpha_2)\left(\frac{\alpha_2\sigma^2\Delta t}{2}\right)\\
&\int_{-\infty}^{\mu_j}\left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(x-(z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t)^2}{2\sigma^2\Delta t}\right)\right)dx
\end{aligned} \tag{4.81}
$$

$$
\begin{aligned}
f_{X+Q}(x) =& p_k\alpha_1\lambda_j^u exp\frac{2\alpha_1\mu_j+\alpha_1\sigma^2\Delta t}{2} exp(-(z-\mu_d\Delta t)\alpha_1)\Phi_a(\mu_j)\\
&+ q_k\alpha_2\lambda_j^d exp\left(\frac{(2\alpha_2\mu_j+\alpha_2\sigma^2\Delta t)}{2}\right) exp((z-\mu_d\Delta t)\alpha_2)\Phi_b(-\mu_j)
\end{aligned} \tag{4.82}
$$

where, $\Phi_a(\mu_j)$ and $1-\Phi_b(\mu_j) = \Phi_b(-\mu_j)$ are the cumulative normal distributions of $x \sim N(x; (z-\mu_d\Delta t)-\alpha_1\sigma^2\Delta t, 2\sigma^2\Delta t)$ and $x \sim Nxy; (z-\mu_d\Delta t)+\alpha_2\sigma^2\Delta t, 2\sigma^2\Delta t)$ respectively at $x = \mu_j$ and $x = -\mu_j$

Putting equation (4.82) into equation (4.71) with $\mu_d = \mu - \frac{1}{2}\sigma^2$ gives the desired result.

4.6.3 The modified double Rayleigh jump-diffusion model

Here, a new model which belongs to the family of the skewed jump-diffusion models called the modified Double Rayleigh jump diffusion (MDRJD) model for stock price process was proposed. The distribution of the upward and downward jump processes are assumed to obey the extended Rayleigh distribution of two positive parameters (location and scale parameters) with probabilities $p, q \geq 0, p+q = 1$. In this case, the jump amplitude is driven by the modified (two-parameter) double Rayleigh distribution given in the definition below:

Definition 4.8: Probability density function of the modified double Rayleigh random variable

Let Y be a Rayleigh random variable, then the Probability density function of the modified double Rayleigh random variable is given as:

$$f_Y(y) = p\frac{y-u}{b^2}exp\left(\frac{-(y-\mu)^2}{2b}\right)1_{[\mu,\infty)}(y) - q\frac{y-u}{a^2}exp\left(\frac{-(y-\mu)^2}{2a}\right)1_{(-\infty,\mu)}(y) \tag{4.83}$$

where $a, b, \mu > 0$ $p, q \geq 0$ and $p + q = 1$

4.6.4 The density of the modified double Rayleigh jump-diffusion process

Now, let $\tilde{S}_t$ be the price process of the stock, which satisfies the Markov process defined on a filtered probability space $(\Omega, F, \mathbb{P}, \mathbb{F})$. Consider that the dynamics of $\tilde{S}_t$ is given as:

$$d\tilde{S}_t = \mu\tilde{S}_t dt + \sigma\tilde{S}_t dW_t + \tilde{S}_t J(Q_j)dN_t \tag{4.84}$$

where, $S_0 \geq 0$, μ the mean return rate of the diffusive process, σ is the diffusive volatility, W_t is a standard Brownian motion, N_t is a Poisson process with respect to the filtration $\mathbb{F}$ having a constant jump rate λ; and $J(Q_j)$ is a non-constant jump amplitude (random jump process), where also, the random variables W_t, N_t and $J(Q_j)$ are independent. The random jump process above can be expressed as:

$$\int_{\zeta_t}^{\zeta_{t+\Delta t}} J(Q_j)dN_t = \sum_{j=1}^{\Delta N_t} J(Q_j), \qquad \zeta_{t+\Delta t} - \zeta_t = \Delta N_t \tag{4.85}$$

given that the $Q_j's$ are i.i.d random variables, the probability density function of N_t is:

$$\mathbb{P}(N_t = k) = \frac{(\lambda\Delta t)^k exp(-\lambda\Delta t)}{k!} \tag{4.86}$$

The solution of equation (4.84) can be obtained by the Itô's formula for jump diffusion and the log returns is obtained as:

$$\Delta(\ln \tilde{S}_t) = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma \Delta W_t + J(Q_j)\Delta N_t \tag{4.87}$$

Next, the probability density function of the process in equation (4.87) subject to the fact that the $Q_j's$ satisfies equation (4.83) was determined. In this model, the jump amplitude Q_j in equation (4.87) was separated into two: the upward random jump process Q_j^u with probability p and the downward random jump process Q_j^d with probability q. The processes are assumed to be the modified Rayleigh distributions of two parameters: $Q_j^u \sim MDR(\mu_j, \sigma_j^u)$ and $Q_j^d \sim MDR(\mu_j, \sigma_j^d)$ respectively, given as the upward and downward jump processes with mean rates and volatilities. It is also assumed that Q_j^u and Q_j^d are independent and identically distributed (i.i.d) random variables. The choice of the MDR distribution for the random jump process is to give a generalisation of a jump-diffusion model which is skewed with non-zero location parameters for both the upward and downward jump processes.

Owing to the above-mentioned, the probability density function of Q under the MDRJD model was defined as:

$$\begin{aligned} f_Q(y) =& p\frac{(y-\mu_j)}{(\sigma_j^u)^2} exp\left(\frac{-(y-\mu_j)^2}{2\sigma_j^u}\right) 1_{[\mu_j,\infty)}(y) \\ &- q\frac{(y-\mu_j)}{(\sigma_j^d)^2} exp\left(\frac{-(y-\mu_j)^2}{2\sigma_j^d}\right) 1_{(-\infty,\mu_j)}(y) \end{aligned} \tag{4.88}$$

In the next Theorem, the probability density function of the log returns of the jump-diffusion process when Q satisfies equation (4.88) was derived.

Theorem 4.2

The probability density for the MDR jump-diffusion process is given by

$$\begin{aligned} f_{\Delta(\ln \tilde{S}_t)x} =& \frac{(1-\lambda\Delta t)}{\sigma\sqrt{\Delta t}} \varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) + \left(p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2} exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)\right.\right. \\ &\left. + \theta\sqrt{\pi\vartheta}\Phi_a(\mu_j) - \mu_j\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right) - q\hat{\eta} exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2} exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)\right. \\ &\left.\left. + \theta\sqrt{\pi\hat{\vartheta}}\Phi_b(\mu_j) - \mu_j\sqrt{\pi\hat{\vartheta}}\Phi_b(-\mu_j)\right)\right)\Delta t \end{aligned} \tag{4.89}$$

where, $\theta = \frac{\sigma_j^u(\mu_d \Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}$, $\rho = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}$, $\vartheta = \frac{2\sigma^2\Delta t\sigma_j^u}{(\sigma_j^u+\sigma^2\Delta t)}$, $\hat{\theta} = \frac{\sigma_j^d(\mu_d \Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$, $\hat{\rho} = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$, $\hat{\vartheta} = \frac{2\sigma^2\Delta t\sigma_j^d}{(\sigma_j^d+\sigma^2\Delta t)}$, $\eta = \frac{\lambda_j^u}{(\sigma_j^u)}\varphi\left(\frac{(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$ and $\hat{\eta} = \frac{\lambda_j^d}{(\sigma_j^d)}\varphi\left(\frac{(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$

Proof.:

Let $\tilde{S}_t$ be the stock price process defined on $(\Omega, F, \mathbb{P}, \mathbb{F})$ such that the dynamics of $\tilde{S}_t$ is given by:

$$d\tilde{S}_t = \mu\tilde{S}_t dt + \sigma\tilde{S}_t dW_t + \tilde{S}_t J(Q_j)dN_t \tag{4.90}$$

where, $S_0 > 0$, $\mu_d = \mu - \frac{1}{2}\sigma^2$, σ, W_t are as previously defined. The random jump process $J(Q_j)$ in equation 4.90 above, is given as:

$$\int_{\zeta_t}^{\zeta_{t+\Delta t}} J(Q_j)dN_t = \sum_{j=1}^{\Delta N_t} J(Q_j) \tag{4.91}$$

where $N_{t(}t \geq 0)$ is a Poisson process with intensity rate λ and density

$$\mathbb{P}(N_t = k) = \frac{(\lambda\Delta t)^k exp(-\lambda\Delta t)}{k!} \tag{4.92}$$

By the $It\hat{\hat{o}}'s$ lemma, the dynamics in equation (4.90) can be obtained and given as equation (4.59):

Also, owing to the condition given above for $J(Q_j)$,

$$\Delta(\ln\tilde{S}_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\Delta W_t + J(Q_j^u)\Delta N_t^u + J(Q_j^d)\Delta N_t^d \tag{4.93}$$

This implies $\tilde{S}_t = S_0 exp((\mu - \frac{1}{2}\sigma^2))\Delta t + \sigma\Delta W_t + \sum_{i=1}^{\Delta N_t^u} Q_j^u + \sum_{i=1}^{\Delta N_t^d} Q_j^d$

Let's assume that the random jump process satisfies equation (4.70):

Then, the density of the process in equation (4.93), can be obtained via:

$$f_{\Delta(\ln\tilde{S}_t)}(x) = (1 - \lambda\Delta t)f_{X_t}(x) + \lambda\Delta t f_{X_t+Q_t}(x) \tag{4.94}$$

From the above, given that

$$X_t = (\mu - \frac{1}{2}\sigma^2)\Delta t + \sigma\Delta W_t$$

and

$$Q_t = \sum_{i=1}^{\Delta N_t^u} Q_j^u + \sum_{i=1}^{\Delta N_t^d} Q_j^d$$

Then,

$$f_{X_t}(x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\Big(- \frac{(x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\Big) = \varphi\Big(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\Big) \tag{4.95}$$

and

$$\begin{aligned} f_Q(x) =& p\lambda_j^u \frac{(x-\mu_j)}{(\sigma_j^u)^2} exp\Big(\frac{-(x-\mu_j)^2}{2\sigma_j^u}\Big) 1_{[\mu_j,\infty)}(x) \\ & - q\lambda_j^d \frac{(x-\mu_j)}{(\sigma_j^d)^2} exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^d}\right) 1_{(-\infty,\mu_j)}(x) \end{aligned} \tag{4.96}$$

In equation (4.94) above, the density $f_{X_t}(x)$ is known already, and $f_{X_t+Q_t}(x)$ was determined by the convolution of densities, it follows for $z \in \mathbb{R}$ that:

$$f_{X+Q}(x) = G(x) = \int_{-\infty}^{\infty} f_X(z-x) f_Q(x) dx \tag{4.97}$$

$$\begin{aligned} = \int_{-\infty}^{\infty} & \left(\frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) \left(p\lambda_j^u \frac{(x-\mu_j)}{(\sigma_j^u)^2}\right.\right. \\ & \left.\left. exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^u}\right) 1_{[\mu_j,\infty)}(x) - q\lambda_j^d \frac{(x-\mu_j)}{(\sigma_j^d)^2} exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^d}\right) 1_{(-\infty,\mu_j)}(x)\right)\right) dx \end{aligned}$$

$$\begin{aligned} G(x) =& \int_{-\mu_j}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) p\lambda_j^u \frac{(x-\mu_j)}{(\sigma_j^u)^2} exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^u}\right) dx \\ & - \int_{-\infty}^{\mu_j} \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} exp\left(-\frac{(z-x-(\mu-\frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) q\lambda_j^d \frac{(x-\mu_j)}{(\sigma_j^d)^2} exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^d}\right) dx \end{aligned} \tag{4.98}$$

For simplicity, let the diffusive drift be $\mu_d = \mu - \frac{1}{2}\sigma^2$, then equation (4.98) can be rewritten as:

$$G(x) =\frac{p\lambda_j^u}{(\sigma_j^u)^2}\int_{-\mu_j}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\left(-\frac{(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right)exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^u}\right)dx$$
$$-\frac{q\lambda_j^d}{(\sigma_j^d)^2}\int_{-\infty}^{\mu_j}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\left(-\frac{(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right)exp\left(\frac{-(x-\mu_j)^2}{2\sigma_j^d}\right)dx \quad (4.99)$$

$$=\frac{p\lambda_j^u}{(\sigma_j^u)^2}\int_{-\mu_j}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\left(-\frac{(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^u}\right)dx$$
$$-\frac{q\lambda_j^d}{(\sigma_j^d)^2}\int_{-\infty}^{\mu_j}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\left(-\frac{(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^d}\right)dx$$

$$=\frac{p\lambda_j^u}{(\sigma_j^u)^2}\int_{-\mu_j}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\,(V)\,dx-\frac{q\lambda_j^d}{(\sigma_j^d)^2}\int_{-\infty}^{\mu_j}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}(x-\mu_j)exp\,(U)\,dx$$

where,

$$V=-\frac{x^2-2x(z-x-\mu_d\Delta t)-(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^u}$$

and

$$U=-\frac{x^2-2x(z-x-\mu_d\Delta t)(z-x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^d}$$

Now, for $\wedge_1=-\frac{x^2-2x(z-\mu_d\Delta t)}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^u}$ and $\wedge_2=-\frac{x^2-2x(z-\mu_d\Delta t)}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^d}$, we have

$$G(x) =\frac{p\lambda_j^u}{(\sigma_j^u)^2}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}exp\left(\frac{-(x-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right)\int_{-\mu_j}^{\infty}(x-\mu_j)exp\,(\wedge_1)\,dx$$
$$-\frac{q\lambda_j^d}{(\sigma_j^d)^2}\frac{1}{\sqrt{2\pi\sigma^2\Delta t}}exp\left(\frac{-(z-\mu_d\Delta t)^2}{2\sigma^2\Delta t}\right)\int_{-\infty}^{\mu_j}(x-\mu_j)exp\,(\wedge_2)\,dx$$

$$G(x) =\frac{p\lambda_j^u}{(\sigma_j^u)^2}\varphi\left(\frac{z-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)\int_{-\mu_j}^{\infty}(z-\mu_j)exp\,(\wedge_1)\,dx$$
$$-\frac{q\lambda_j^d}{(\sigma_j^d)^2}\varphi\left(\frac{z-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)\int_{-\infty}^{\mu_j}(x-\mu_j)exp\,(\wedge_2)\,dx \quad (4.100)$$

Let, $p\eta=\frac{p\lambda_j^u}{(\sigma_j^u)^2}\varphi\left(\frac{z-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$ and $q\hat{\eta}=\frac{q\lambda_j^d}{(\sigma_j^d)^2}\varphi\left(\frac{z-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)$ in equation (4.100), then we have:

$$
\begin{aligned}
G(x) =& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{x^2-2x(z-\mu_d\Delta t)}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^u}\right)dx \\
&- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{x^2-2x(z-\mu_d\Delta t)}{2\sigma^2\Delta t}-\frac{(x-\mu_j)^2}{2\sigma_j^d}\right)dx
\end{aligned}
\tag{4.101}
$$

$$
\begin{aligned}
=& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{\sigma_j^u(x^2-2x(z-\mu_d\Delta t))+\sigma^2\Delta t(x-\mu_j)^2}{2\sigma^2\Delta t\sigma_j^u}\right)dx \\
&- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{\sigma_j^d(x^2-2x(z-\mu_d\Delta t))+\sigma^2\Delta t(x-\mu_j)^2}{2\sigma^2\Delta t\sigma_j^d}\right)dx
\end{aligned}
\tag{4.102}
$$

$$
\begin{aligned}
=& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{\sigma_j^u x^2-2x\sigma_j^u(z-\mu_d\Delta t)+x^2\sigma^2\Delta t-2x\mu_j\sigma^2\Delta t+\mu_j^2\sigma^2\Delta t}{2\sigma^2\Delta t\sigma_j^u}\right)dx \\
&- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{\sigma_j^d x^2-2x\sigma_j^d(z-\mu_d\Delta t)+x^2\sigma^2\Delta t-2x\mu_j\sigma^2\Delta t+\mu_j^2\sigma^2\Delta t}{2\sigma^2\Delta t\sigma_j^d}\right)dx
\end{aligned}
$$

$$
\begin{aligned}
=& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{(\sigma_j^u+\sigma^2\Delta t)x^2-2y(\sigma_j^u(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)+\mu_j^2\sigma^2\Delta t}{2\sigma^2\Delta t\sigma_j^u}\right)dx \\
&- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{(\sigma_j^d+\sigma^2\Delta t)x^2-2y(\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)+\mu_j^2\sigma^2\Delta t}{2\sigma^2\Delta t\sigma_j^d}\right)dy
\end{aligned}
$$

$$
\begin{aligned}
=& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{x^2-2y\frac{(\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^u+\sigma^2\Delta t)}+\frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}}{\frac{2\sigma^2\Delta t\sigma_j^u}{(\sigma_j^u+\sigma^2\Delta t)}}\right)dx \\
&- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{x^2-2x\frac{(\sigma_j^d(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^u+\sigma^2\Delta t)}+\frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}}{\frac{2\sigma^2\Delta t\sigma_j^d}{(\sigma_j^d+\sigma^2\Delta t)}}\right)dx
\end{aligned}
$$

For.

$$
\Theta_1 = -\frac{\left(x-\frac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2-\left(\frac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2+\frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}}{\frac{2\sigma^2\Delta t\sigma_j^u}{(\sigma_j^u+\sigma^2\Delta t)}}
$$

and

$$\Theta_2 = -\frac{\left(x - \frac{\sigma_j^d(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}\right)^2 - \left(\frac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^d+\sigma^2\Delta t)}\right)^2 + \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}}{\frac{2\sigma^2\Delta t\sigma_j^d}{(\sigma_j^d+\sigma^2\Delta t)}}$$

Then,

$$G(x) = p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\,(\Theta_1)\,dx - q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\,(\Theta_2)\,dx \tag{4.103}$$

Let, $\theta = \frac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t)}{(\sigma_j^u+\sigma^2\Delta t)}, \rho = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}, \vartheta = \frac{2\sigma^2\Delta t\sigma_j^u}{(\sigma_j^u+\sigma^2\Delta t)}$
$\hat{\theta} = \frac{\sigma_j^d(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}, \hat{\rho} = \frac{\mu_j^2\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$ and $\hat{\varphi} = \frac{2\sigma^2\Delta t\sigma_j^d}{(\sigma_j^d+\sigma^2\Delta t)}$ in equation (4.103)

Then,

$$\begin{aligned} G(x) =& p\eta\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{\left(x-\theta\right)^2-\left(\theta\right)^2+\rho}{\vartheta}\right)dx \\ &- q\hat{\eta}\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{\left(x-\hat{\theta}\right)^2-\left(\hat{\theta}\right)^2+\hat{\rho}}{\hat{\vartheta}}\right)dx \end{aligned} \tag{4.104}$$

$$\begin{aligned} =& p\eta exp\left(\frac{\theta^2}{\vartheta}\right)exp\left(\frac{-\rho}{\vartheta}\right)\int_{-\mu_j}^{\infty}(x-\mu_j)exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx \\ &- q\hat{\eta}exp\left(\frac{\hat{\theta}^2}{\hat{\vartheta}}\right)exp\left(\frac{-\hat{\rho}}{\hat{\vartheta}}\right)\int_{-\infty}^{\mu_j}(x-\mu_j)exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx \end{aligned}$$

$$\begin{aligned} =& p\eta exp\left(\frac{\theta^2}{\vartheta}\right)exp\left(\frac{-\rho}{\vartheta}\right)\left(\int_{\mu_j}^{\infty}xexp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx - \mu_j\int_{\mu_j}^{\infty}exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx\right. \\ &- q\hat{\eta}exp\left(\frac{\hat{\theta}^2}{\hat{\vartheta}}\right)exp\left(\frac{-\hat{\rho}}{\hat{\vartheta}}\right)\int_{-\infty}^{\mu_j}yexp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx - \mu_j\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx \end{aligned}$$

Let, $x = (x-\theta)+\theta$ and $x = (x-\hat{\theta})+\hat{\theta}$ be used in the first and third integrals respectively above, we get:

$$
\begin{aligned}
G(x) =& p\eta exp\left(\frac{\theta^2}{\vartheta}\right) exp\left(\frac{-\rho}{\vartheta}\right)\Bigg(\int_{\mu_j}^{\infty}((x-\theta)+\theta)exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx\\
&-\mu_j\int_{\mu_j}^{\infty}exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx - q\hat{\eta}exp\left(\frac{\hat{\theta}^2}{\hat{\vartheta}}\right)exp\left(\frac{-\hat{\rho}}{\hat{\vartheta}}\right)\\
&\Bigg(\int_{-\infty}^{\mu_j}\Big((x-\hat{\theta}+\hat{\theta}\Big)exp\left(-\frac{\left(y-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx - \mu_j\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
=& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\Bigg(\int_{\mu_j}^{\infty}(x-\theta)exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx\\
&+\theta\int_{\mu_j}^{\infty}exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx - \mu_j\int_{\mu_j}^{\infty}exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx\Bigg)\\
&-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\Bigg(\int_{-\infty}^{\mu_j}\Big((x-\hat{\theta}\Big)exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx\\
&+\hat{\theta}\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx - \mu_j\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx\Bigg)
\end{aligned}
\tag{4.105}
$$

Applying integration by substitution in equation (4.105) to the 1st and 4th integral, let $u = \frac{(x-\theta)^2}{\vartheta} \implies dx = \frac{\vartheta}{2(x-\theta)}du$ and $v = -\frac{(x-\hat{\theta})^2}{\hat{\vartheta}} \implies dx = \frac{-\hat{\vartheta}}{2(x-\hat{\theta})}dv$ respectively.

Then, equation (4.105) gives the expression below:

$$
\begin{aligned}
G(x) =& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\Bigg(\int_{\frac{(\mu_j-\theta)^2}{\vartheta}}^{\infty}(x-\theta)exp\left(-u\right)\frac{\vartheta}{2(y-\theta)}du\\
&+\theta\int_{\mu_j}^{\infty}exp\left(-\frac{\left(x-\theta\right)^2}{\vartheta}\right)dx - \mu_j\int_{\mu_j}^{\infty}exp\left(-\frac{\left(y-\theta\right)^2}{\vartheta}\right)dx\Bigg)\\
&-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\Bigg(\int_{-\infty}^{-\frac{(\mu_j-\theta)^2}{\vartheta}}\Big((x-\hat{\theta}\Big)exp\left(v\right)\frac{-\hat{\vartheta}}{2(y-\hat{\theta})}dv\\
&+\hat{\theta}\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx - \mu_j\int_{-\infty}^{\mu_j}exp\left(-\frac{\left(x-\hat{\theta}\right)^2}{\hat{\vartheta}}\right)dx\Bigg)
\end{aligned}
\tag{4.106}
$$

$$
\begin{aligned}
G(x) =& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\int_{\mu_j}^{\infty}exp\left(-\frac{\big(x-\theta\big)^2}{\vartheta}\right)dy\right. \\
&\left.-\mu_j\int_{\mu_j}^{\infty}exp\left(-\frac{\big(x-\theta\big)^2}{\vartheta}\right)dx\right)-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)\right. \\
&\left.+\hat{\theta}\int_{-\infty}^{\mu_j}exp\left(-\frac{\big(x-\hat{\theta}\big)^2}{\hat{\vartheta}}\right)dx-\mu_j\int_{-\infty}^{\mu_j}exp\left(-\frac{\big(x-\hat{\theta}\big)^2}{\hat{\vartheta}}\right)dx\right)
\end{aligned}
\tag{4.107}
$$

In equation (4.107),

$$
exp\left(-\frac{\big(x-\theta\big)^2}{\vartheta}\right)=exp\left(-\frac{\left(x-\dfrac{\sigma_j^u(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\right)
\tag{4.108}
$$

and

$$
exp\left(-\frac{\big(x-\hat{\theta}\big)^2}{\hat{\vartheta}}\right)=exp\left(-\frac{\left(x-\dfrac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^d+\sigma^2\Delta t}}\right)
\tag{4.109}
$$

Recall that $N(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)$

Thus,

$$
\begin{aligned}
&exp\left(-\frac{\left(x-\dfrac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\right) \\
&=\frac{\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}}{\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}}exp\left(-\frac{\left(x-\dfrac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\right) \\
&=\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\boldsymbol{N}\left(x,\frac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)},\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}\right)
\end{aligned}
\tag{4.110}
$$

where, $\boldsymbol{N}\left(x, \dfrac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}, \frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}\right)$ is a normal probability density function with mean $\dfrac{\sigma_j^u(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}$ and variance $\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}$

Similarly, in equation (4.107),

$$
\begin{aligned}
&exp\left(-\frac{\left(x-\dfrac{\sigma_j^d(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^d}{\sigma_j^u+\sigma^2\Delta t}}\right) \\
&=\frac{\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}}{\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^d}{\sigma_j^u+\sigma^2\Delta t}}}exp\left(-\frac{\left(x-\dfrac{\sigma_j^d(z-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}\right)^2}{\dfrac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\right) \qquad (4.111)\\
&=\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\boldsymbol{N}\left(x; \frac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}, \frac{2\sigma^2\Delta t\sigma_j^d}{\sigma_j^d+\sigma^2\Delta t}\right),
\end{aligned}
$$

where, $\boldsymbol{N}\left(x; \dfrac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}, \frac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^d+\sigma^2\Delta t}\right)$ is a normal probability density function with mean $\dfrac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^d+\sigma^2\Delta t)}$ and variance $\frac{2\sigma^2\Delta t\sigma_j^u}{\sigma_j^d+\sigma^2\Delta t}$

But

$$
\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\boldsymbol{N}\left(x; \frac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}, \frac{2\sigma^2\Delta t\sigma_j^d}{\sigma_j^d+\sigma^2\Delta t}\right)=\sqrt{\pi\vartheta}\boldsymbol{N}(x;\theta,\vartheta) \qquad (4.112)
$$

and

$$
\sqrt{\frac{2\pi\sigma^2\Delta t\sigma_j^u}{\sigma_j^u+\sigma^2\Delta t}}\boldsymbol{N}\left(x; \frac{\sigma_j^d(x-\mu_d\Delta t)+\mu_j\sigma^2\Delta t}{(\sigma_j^u+\sigma^2\Delta t)}, \frac{2\sigma^2\Delta t\sigma_j^d}{\sigma_j^d+\sigma^2\Delta t}\right)=\sqrt{\pi\hat{\vartheta}}\boldsymbol{N}(x;\hat{\theta},\hat{\vartheta}) \qquad (4.113)
$$

Substituting equations (4.112) and (4.113) into equation (4.111), gives:

$$
\begin{aligned}
G(x) =& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\sqrt{\pi\vartheta}\int_{\mu_j}^{\infty}\boldsymbol{N}(x;\theta,\vartheta)dx\right.\\
&\left.-\mu_j\sqrt{\pi\vartheta}\int_{\mu_j}^{\infty}\boldsymbol{N}(x;\theta,\vartheta)dx\right)-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)\right.\\
&\left.+\hat{\theta}\sqrt{\pi\hat{\vartheta}}\int_{-\infty}^{\mu_j}\boldsymbol{N}(x;\hat{\theta},\hat{\vartheta})dx-\mu_j\sqrt{\pi\hat{\vartheta}}\int_{-\infty}^{\mu_j}\boldsymbol{N}(x;\hat{\theta},\hat{\vartheta})dx\right)
\end{aligned}
$$

$$
\begin{aligned}
=& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\sqrt{\pi\vartheta}\Phi_a(\mu_j)-\mu_j\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right)\\
&-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)+\hat{\theta}\sqrt{\pi\hat{\vartheta}}\Phi_b(\mu_j)-\mu_j\sqrt{\pi\hat{\vartheta}}\Phi_b(-\mu_j)\right)
\end{aligned}
\tag{4.114}
$$

where, $\Phi_a(\mu_j)$ and $1-\Phi_b(\mu_j)$ are the cumulative normal distribution of $x \sim \boldsymbol{N(x;\theta,\vartheta)}$ and $x \sim \boldsymbol{N(x;\hat{\theta},\hat{\vartheta})}$ respectively at $x=\mu_j$ and $x=-\mu_j$

Thus, it followed from equation (4.114) that

$$
\begin{aligned}
f_{X_{t+Q_t}} =& p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\sqrt{\pi\vartheta}\Phi_a(\mu_j)-\mu_j\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right)\\
&-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)+\theta\sqrt{\pi\hat{\vartheta}}\Phi_b(\mu_j)-\mu_j\sqrt{\pi\hat{\vartheta}}\Phi_b(-\mu_j)\right)
\end{aligned}
$$

Hence, the probability density function of the MDRJD model was obtained as:

$$
\begin{aligned}
&f_{\Delta(\ln\tilde{S}_t)}(x)=\frac{(1-\lambda\Delta t)}{\sigma\sqrt{\Delta t}}\varphi\left(\frac{x-(\mu-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right)\\
&+\left(p\eta exp\left(\frac{\theta^2-\rho}{\vartheta}\right)\left(\frac{\theta}{2}exp\left(-\frac{(\mu_j-\theta)^2}{\vartheta}\right)+\theta\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right.\right.\\
&\left.-\mu_j\sqrt{\pi\vartheta}\Phi_a(\mu_j)\right)-q\hat{\eta}exp\left(\frac{\hat{\theta}^2-\hat{\rho}}{\hat{\vartheta}}\right)\left(\frac{\hat{\vartheta}}{2}exp\left(-\frac{(\mu_j-\hat{\theta})^2}{\hat{\vartheta}}\right)\right.\\
&\left.\left.+\theta\sqrt{\pi\hat{\vartheta}}\Phi_b(\mu_j)-\mu_j\sqrt{\pi\hat{\vartheta}}\Phi_b(-\mu_j)\right)\right)\Delta t
\end{aligned}
$$

4.6.5 The Lévy-Khintchine formulae for the ALJD and the MDRJD processes

Next, the Lévy-Khintchine formulae for the AL and MDR jiump-diffusion processes was derived.

Theorem 4.4

The Lévy-Khintchine (LK) formula of the ALJD process is given as:

$$\psi^{aljd}(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 - \left(\frac{\lambda_j^u p_k \alpha_1}{\alpha_1 - iu} + \frac{\lambda_j^d q_k \alpha_2}{\alpha_2 + iu}\right)\mathrm{e}^{iu\mu_j} + \lambda_j^d q_k + \lambda_j^u p_k \tag{4.115}$$

where, $\mu \in \mathbb{R}$ and $\sigma > 0$ are respectively the diffusive drift and volatility, $p_k, q_k > 0$ controls the skewness of the jump measure and λ_j^u, λ_j^d are the jump-intensities of the upward and downward jump processes.

Proof.:

Recall that by for a Lévy- Process X_t, given its Lévy triple as $(\mu, \sigma^2, v(dx))$; an expression for the characteristic exponent $\psi(\mu) = log\phi_{X_t}(u), u \in \mathbb{R}$ where $\phi_{X_t}(u)$ is the characteristic function of X_t.

Then,

$$\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux1_{\{|x|\leq 1\}})v(dx) \tag{4.116}$$

In this case, by definition, the ALJD process has finite number of jumps in a finite period of time. Hence,

$$\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\Re} (\mathrm{e}^{iux} - 1)\nu(dx) \tag{4.117}$$

where, $\quad \nu(dx) = \lambda f(x)$

Recall from equation (4.117) above that the Lévy-density or measure of the jump size of the ALJD process is given as:

$$\begin{aligned} \lambda f(x) &= \lambda_j^u p_k \alpha_1 exp(-\alpha_1(x - \mu_j))1_{[\mu_j,\ \infty]}(x) \\ &+ \lambda_j^d q_k \alpha_2 exp(\alpha_2(x - \mu_j))1_{(-\infty,\ \mu_j)}(x) \end{aligned} \tag{4.118}$$

According to equation (4.117),

$$\psi(u) = iu\mu - \frac{1}{2}\sigma^2u^2 + \int_{\mathbb{R}}\Bigg((\mathrm{e}^{iux}-1)\Bigg(\lambda_j^u p_k\alpha_1 exp(-\alpha_1(x-\mu_j))1_{[\mu_j,\infty]}(x)$$
$$+ \lambda q_k\alpha_2 exp(\alpha_2(x-\mu_j))1_{(-\infty,\mu_j)}(x)\Bigg)\Bigg)dx$$

$$= iu\mu - \frac{1}{2}\sigma^2u^2 + \int_{\mu_j}^{\infty}(\mathrm{e}^{iux}-1)\lambda_j^u p_k\alpha_1\mathrm{e}^{-\alpha_1(x-\mu_j)}dx$$
$$+ \int_{-\infty}^{\mu_j}(\mathrm{e}^{iux}-1)\lambda_j^d q_k\alpha_2\mathrm{e}^{\alpha_2(x-\mu_j)}dx$$

$$=iu\mu - \frac{1}{2}\sigma^2u^2 + \lambda_j^u p_k\alpha_1\int_{\mu_j}^{\infty}\left(\mathrm{e}^{iux}\mathrm{e}^{-\alpha_1(x-\mu_j)} - \mathrm{e}^{-\alpha_1(x-\mu_j)}\right)dx$$
$$+ \lambda_j^d q_k\alpha_2\int_{-\infty}^{\mu_j}\left(\mathrm{e}^{iux}\mathrm{e}^{-\alpha_2(x-\mu_j)} - \mathrm{e}^{-\alpha_2(x-\mu_j)}\right)dx$$

$$=iu\mu - \frac{1}{2}\sigma^2u^2 + \lambda_j^u p_k\alpha_1\int_{\mu_j}^{\infty}\left(\mathrm{e}^{-(\alpha_1-iu)x+\alpha_1\mu_j} - \mathrm{e}^{-\alpha_1x+\alpha_1\mu_j)}\right)dx$$
$$+ \lambda_j^d q_k\alpha_2\int_{-\infty}^{\mu_j}\left(\mathrm{e}^{(\alpha_2+iu)x-\alpha_2\mu_j} - \mathrm{e}^{\alpha_2x-\alpha_2\mu_j}\right)dx$$

$$=iu\mu - \frac{1}{2}\sigma^2u^2 + \lambda_j^u p_k\alpha_1\left(\frac{-1}{\alpha_1-iu}\mathrm{e}^{-(\alpha_1-iu)x+\alpha_1\mu_j} + \frac{1}{\alpha_1}\mathrm{e}^{-\alpha_1x+\alpha_1\mu_j}\Bigg|_{\mu_j}^{\infty}\right)$$
$$+ \lambda_j^d q_k\alpha_2\left(\frac{-1}{\alpha_2+iu}\mathrm{e}^{(\alpha_2+iu)x-\alpha_2\mu_j} - \frac{1}{\alpha_2}\mathrm{e}^{\alpha_2x-\alpha_2\mu_j}\Bigg|_{-\infty}^{\mu_j}\right)$$

$$=iu\mu - \frac{1}{2}\sigma^2u^2 + \lambda_j^u p_k\alpha_1\left(\frac{-1}{\alpha_1-iu}\mathrm{e}^{-\alpha_1-iu\mu_j+\alpha_1\mu_j} + \frac{1}{\alpha_1}\mathrm{e}^{-\alpha_1\mu_j+\alpha_1\mu_j}\Bigg|_{\mu_j}^{\infty}\right)$$
$$+ \lambda_j^d q_k\alpha_2\left(-\left(\frac{1}{\alpha_2+iu}\mathrm{e}^{\alpha_2\mu_j+iu\mu_j-\alpha_2\mu_j} - \frac{1}{\alpha_2}\mathrm{e}^{-\alpha_2\mu_j-\alpha_2\mu_j}\right)\right)$$

$$= iu\mu - \frac{1}{2}\sigma^2u^2 + \lambda_j^u p_k\alpha_1\left(\frac{-\mathrm{e}^{iu\mu_j}}{\alpha_1-iu} + \frac{1}{\alpha_1}\right) + \lambda_j^d q_k\alpha_2\left(\frac{-\mathrm{e}^{iu\mu_j}}{\alpha_2+iu} + \frac{1}{\alpha_2}\right)$$

$$= iu\mu - \frac{1}{2}\sigma^2u^2 - \frac{\lambda_j^u p_k\alpha_1\mathrm{e}^{iu\mu_j}}{\alpha_1-iu} + \lambda_j^u p_k - \frac{\lambda_j^d q_k\alpha_2\mathrm{e}^{iu\mu_j}}{\alpha_2+iu} + \lambda_j^d q_k$$

Therefore, the characteristic exponent obtained from the Lévy-Khintchine Theorem for the ALJD process is given as:

$$\psi^{aljd}(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 - \left(\frac{\lambda_j^u p_k \alpha_1}{\alpha_1 - iu} + \frac{\lambda_j^d q_k \alpha_2}{\alpha_2 + iu}\right)\mathrm{e}^{iu\mu_j} + \lambda_j^d q_k + \lambda_j^u p_k \tag{4.119}$$

Theorem 4.3

Given a jump measure $\lambda f_Q(x)$, of the MDRJD process, such that:

$$\lambda f_Q(x) = p\lambda_j^u\left(\frac{x-\mu_j}{(\sigma_j^u)^2}\right)e^{-\frac{(x-\mu_j)^2}{2\sigma_j^u}} 1_{[\mu_j,\infty)}(x) + q\lambda_j^d\left(\frac{x-\mu_j}{(\sigma_j^d)^2}\right)e^{-\frac{(x-\mu_j)^2}{2\sigma_j^d}} 1_{[-\infty,\mu_j)}(x) \tag{4.120}$$

Then, the LK formula for the MDRJD process is:

$$\psi^{mdrjd}(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 - \frac{p\lambda_j^u}{\sigma_j^u} + \frac{p\lambda_j^d}{\sigma_j^d} + \left(\frac{p\lambda_j^u}{\sigma_j^u} - \frac{p\lambda_j^d}{\sigma_j^d}\right)e^{iu\mu_j} \tag{4.121}$$

Proof.:

By the Lévy-Itô decomposition of a Lévy process,

$$\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}(\mathrm{e}^{iux} - 1 - iux1_{\{|x|\le 1\}})\nu(dx) \tag{4.122}$$

By definition, $n(\Delta(X_t)) < \infty$ in $[0,t]$ in the MDRJD process. Hence,

$$\psi(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}(\mathrm{e}^{iux} - 1)\nu(dx) \tag{4.123}$$

where, $\nu(dx) = \lambda f(x)$

Note that the Lévy-density or jump measure of the MDRJD process satisfies equation (4.120).

Then, it follows from equations (4.120) and (4.123) that:

$$\begin{aligned}\psi^{mdrjd}(u) = iu\mu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}(\mathrm{e}^{iux} - 1)\Bigg(p\lambda_j^u\frac{x-\mu_j}{(\sigma_j^u)^2}e^{-\frac{(x-\mu_j)^2}{2\sigma_j^u}} 1_{[\mu_j,\infty)}(x) \\ + q\lambda_j^d\frac{x-\mu_j}{(\sigma_j^d)^2}e^{-\frac{(x-\mu_j)^2}{2\sigma_j^d}} 1_{[-\infty,\mu_j)}(x)\Bigg)dx\end{aligned} \tag{4.124}$$

$$
\begin{aligned}
\psi^{mdrjd}(u) =& iu\mu - \frac{1}{2}\sigma^2u^2 + \int_{\mu_j}^{\infty}(\mathrm{e}^{iux}-1)p\lambda_j^u\frac{x-\mu_j}{(\sigma_j^u)^2}e^{-\frac{(x-\mu_j)^2}{2\sigma_j^u}}dx \\
&+ \int_{-\infty}^{\mu_j}(\mathrm{e}^{iux}-1)q\lambda_j^d\frac{x-\mu_j}{(\sigma_j^d)^2}e^{-\frac{(x-\mu_j)^2}{2\sigma_j^d}}dx
\end{aligned} \tag{4.125}
$$

$$
\begin{aligned}
=& iu\mu - \frac{1}{2}\sigma^2u^2 + p\lambda_j^u\int_{\mu_j}^{\infty}\frac{x-\mu_j}{(\sigma_j^u)^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\sigma_j^u}} - e^{-\frac{(x-\mu_j)^2}{2\sigma_j^u}}\right)dx \\
&+ q\lambda_j^d\int_{-\infty}^{\mu_j}\frac{x-\mu_j}{(\sigma_j^d)^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\sigma_j^d}} - e^{-\frac{(x-\mu_j)^2}{2\sigma_j^d}}\right)dx
\end{aligned} \tag{4.126}
$$

In equation (4.126), let

$$
I_1 = \int_{\mu_j}^{\infty}\frac{x-\mu_j}{(\sigma_j^u)^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\sigma_j^u}} - e^{-\frac{(x-\mu_j)^2}{2\sigma_j^u}}\right)dx
$$

and

$$
I_2 = \int_{-\infty}^{\mu_j}\frac{x-\mu_j}{(\sigma_j^d)^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\sigma_j^d}} - e^{-\frac{(x-\mu_j)^2}{2\sigma_j^d}}\right)dx
$$

To keep notations simple, let $\mu_j = \alpha$, $\sigma_j^u = \beta$ and $\sigma_j^d = \gamma$.

Then,

$$
\begin{aligned}
I_1 &= \int_{\alpha}^{\infty}\frac{x-\alpha}{\beta^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\beta}} - e^{-\frac{(x-\alpha)^2}{2\beta}}\right)dx \\
&= \lim_{n\to\infty}\int_{\alpha}^{n}\frac{x-\alpha}{\beta^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\beta}} - e^{-\frac{(x-\alpha)^2}{2\beta}}\right)dx \\
&= \frac{\mathrm{e}^{iu\mu_j}-1}{(\sigma_j^u)^2}.\sigma_j^u\left(1 - \lim_{n\to\infty}\left(\mathrm{e}^{\frac{-(\mu_j-n)^2}{2\sigma_j^u}}\right)\right) \\
&= \frac{\mathrm{e}^{iu\mu_j}-1}{\sigma_j^u}
\end{aligned} \tag{4.127}
$$

and

$$
\begin{aligned}
I_2 &= \lim_{n\to\infty}\int_{\alpha}^{n}\frac{x-\alpha}{\beta^2}\left(\mathrm{e}^{iux-\frac{(x-\mu_j)^2}{2\gamma}} - e^{-\frac{(x-\alpha)^2}{2\gamma}}\right)dx \\
&= \frac{\mathrm{e}^{iu\mu_j}-1}{\sigma_j^d}
\end{aligned} \tag{4.128}
$$

Therefore, it follows from equations (4.126), (4.127) and (4.128) that:

$$
\psi^{mdrjd}(u) = iu\mu - \frac{1}{2}\sigma^2u^2 - \frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d} + \left(\frac{p\lambda_j^u}{\sigma_j^u} - \frac{q\lambda_j^d}{\sigma_j^d}\right)e^{iu\mu_j} \tag{4.129}
$$

4.7 STUDY SIX

Parameter Estimation

In this thesis, novel asymmetric jump-diffusion models for the stock indices data were considered. Hence, the the existing models: the GBM, the symmetric NJD and the asymmetric DEJD models were compared with the novel asymmetric jump-diffusion models described in the previous section.

4.7.1 Initial Parameter Estimation in the AL jump-diffusion Model

The parameters' initial values based on the empirical stock indices data were obtained. For $\Delta t = \frac{1}{252}$ (average of 252 trading days in a year), a decision on the occurrence of a jump in the process was based on:

$$X^{j}_{\Delta,t} = \Delta(ln\tilde{S}_t) > \epsilon \tag{4.130}$$

where, $\epsilon > 0$, a threshold for jumps in the plots of the log returns. Hence, the estimate for λ (the intensity of jumps) was given as:

$$\hat{\lambda} = \frac{n(Q_j)}{(n(\Delta(ln\tilde{S}_t)) - 1)\Delta t} = \frac{n(Q_i^u) + n(Q_i^d)}{(n(\Delta(ln\tilde{S}_t)) - 1)\Delta t} \tag{4.131}$$

The initial estimates of the jump intensities for the upward and downward processes were given respectively as:

$$\hat{\lambda}_j^u = \frac{n(Q_i^u)}{(n(\Delta(ln\tilde{S}_t)) - 1)\Delta t} \tag{4.132}$$

and

$$\hat{\lambda}_j^d = \frac{n(Q_i^d)}{(n(\Delta(ln\tilde{S}_t)) - 1)\Delta t} \tag{4.133}$$

The initial estimates of the parameters enable us to find the optimal values for the parameters that can maximize the Log-likelihood function of the models' densities. In the GBM model, when jumps are assumed to be absent in the stock-indices log returns,

$$\begin{aligned}\mathbb{E}\{X^d_{\Delta t}\} &= \mathbb{E}(X^d_{\Delta t}|N_{t+\Delta t} - N_t = 0)\\ &= E\Bigg(\Bigg(\mu - \frac{1}{2}\sigma^2\Bigg)\Bigg)\Delta t + \sigma(W_{t+\Delta t} - W_t) + \sum_{i=1}^{\Delta N_t = 0} Q_j\Bigg)\\ &= \mathbb{E}\Bigg(\Bigg(\mu - \frac{1}{2}\sigma^2\Bigg)\Bigg)\Delta t + \sigma(W_{t+\Delta t} - W_t) + \mathbb{E}\sum_{i=1}^{\Delta N_t = 0} Q_j\Bigg)\end{aligned}$$

$$\mathbb{E}(X^d_{\Delta t}) = \Bigg(\Bigg(\mu - \frac{1}{2}\sigma^2\Bigg)\Bigg)\Delta t \;\text{ since }\; \mathbb{E}(W_{t+\Delta t} - W_t) = 0 \tag{4.134}$$

Also,

$$Var(X^d_{\Delta t}) = Var\Bigg(\Bigg(\mu - \tfrac{1}{2}\sigma^2\Bigg)\Bigg)\Delta t + \sigma(W_{t+\Delta t} - W_t) + \mathbb{E}\textstyle\sum_{i=1}^{\Delta N_t=0} Q_j$$

$$= \sigma\Delta t \tag{4.135}$$

Hence, the initial values of the mean and volatility parameters are respectively:

$$\hat{\mu}_d = \frac{2\mathbb{E}(X^d_{\Delta t}) + Var(X^d_{\Delta t})}{2\Delta t} \tag{4.136}$$

and

$$\hat{\sigma}^2_d = \frac{Var(X^d_{\Delta t})}{\Delta t} \tag{4.137}$$

Owing to the condition stated in equation (4.130) for ϵ, under the diffusion models with jump process, we split the empirical log returns of the stock indices into two sets, namely: sets P and Q, where $P \cap Q = \emptyset$ is defined as:

$$P = \Big\{|X^d_{\Delta t}| : X_{\Delta t} = \Delta(ln\tilde{S}_t) \leq \epsilon; \epsilon > 0\Big\} \tag{4.138}$$

and

$$Q = \Big\{|X^j_{\Delta t}| : X_{\Delta t} = \Delta(ln\tilde{S}_t) > \epsilon; \epsilon > 0\Big\} \tag{4.139}$$

Also, in the asymmetricjump-diffusion models, set Q was splitted into two subsets. That is,

$$Q^u = \Big\{|X^j_{\Delta t}|^u = Q^u_j : X_{\Delta t} = \Delta(ln\tilde{S}_t) > \epsilon; \epsilon > 0\Big\} \tag{4.140}$$

and

$$Q^d = \Big\{|X^j_{\Delta t}|^d = Q^d_j : X_{\Delta t} = \Delta(ln\tilde{S}_t) < -\epsilon; \epsilon > 0\Big\} \tag{4.141}$$

The initial parameters of the the jump process under the symmetric NJD model jump-diffusion given as:

$$\hat{\mu}_j = \mathbb{E}\Big(X^j_{\Delta t}\Big) - \Big(\mu - \frac{\sigma^2}{2}\Big)\Delta t \tag{4.142}$$

and

$$\hat{\sigma}^2_j = Var(X^j_{\Delta t}) - \hat{\sigma}^2_d \Delta t \tag{4.143}$$

In the asymmetric DEJD model, the initial estimates of: p, q, η_1 and η_2 are obtained from the stock indices empirical data set based on:

$$\hat{p} = \frac{n(Q^u_j)}{n(Q^u_j) + n(Q^d_j)}, \qquad \hat{q} = \frac{n(Q^d_j)}{n(Q^u_j) + n(Q^d_j)} \tag{4.144}$$

and

$$\hat{\eta}_1 = (\mathbb{E}(Q^u_j))^{-1}, \qquad \hat{\eta}_2 = (\mathbb{E}(Q^d_j))^{-1} \tag{4.145}$$

4.7.2 Derivation of the basic moments of the ALJD process via the L$\acute{e}$vy-Khintchine (LK) formula

The basic moments of the ALJD process via the n^{th} cumulant of the ALJD process given the LK formula was derived. The n^{th} cumulant of the characteristic exponent $\psi(u)$ is given as:

$$k_n = \frac{1}{i^n}\psi^{(n)}(u)|_{u=0} \tag{4.146}$$

such that:

$$
\begin{aligned}
k_1 = m_1 = \mathbb{E}^{aljd}(\Delta(lnS_t)\\
k_2 = m_2 = \mathcal{V}^{aljd}(\Delta(lnS_t)\\
k_3 = \frac{m_3}{m_2^{3/2}} = \gamma_1^{aljd}(\Delta(lnS_t)\\
k_4 = \frac{m_4}{m_2^2} - 3 = \gamma_2^{aljd}(\Delta(lnS_t)
\end{aligned}
$$

where, $m_n = \frac{1}{i^n}\phi^{(n)}(u)$, $\phi(u)$ for $u \in \mathbb{R}$, is the characteristic function of X_t, $\mathbb{E}, \mathcal{V}, \gamma_1, \gamma_2$ are respectively the mean, variance, skewness and kurtosis of $\Delta(lnS_t)$. It follows from the above, that

$$
\mathbb{E}^{aljd}(\Delta(lnS_t) = (\mu - 1/2\sigma^2)\Delta t - \lambda_j^u\left(\frac{p_k}{\alpha_1} + \mu_j p_k\right) - \lambda_j^d\left(\frac{q_k}{\alpha_2} + \mu_j q_k\right) \tag{4.147}
$$

$$
\begin{aligned}
\mathcal{V}^{aljd}(\Delta(lnS_t) = \sigma_d^2\Delta t - \frac{2\lambda_j^u p_k}{\alpha_1^3} + 2\mu_j\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right)\\
+ \left(\lambda_j^u p_k + \lambda_j^d q_k\right)\mu_j^2
\end{aligned}
\tag{4.148}
$$

$$
\gamma_1^{aljd}(\Delta(lnS_t) = \frac{\hat{M}}{(\mathit{\Theta})^{\frac{3}{2}}} \tag{4.149}
$$

where,

$$
\begin{aligned}
\hat{M} =& \frac{8\lambda_j^u p_k}{\alpha_1^4} + \frac{2\lambda_j^u p_k}{\alpha_1^3}\mu_j\\
&+ 8\mu_j\left(\frac{\lambda_j^u p_k}{\alpha_1^2} + \frac{\lambda_j^d q_k}{\alpha_2^2}\right) + 2\mu_j\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right)\\
&+ \mu_j^2\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right) + \mu_j^3\left(\lambda_j^u p_k + \lambda_j^d q_k\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathit{\Theta} =& \sigma_d^2\Delta t - \frac{2\lambda_j^u p_k}{\alpha_1^3} + 2\mu_j\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right)\\
&+ \left(\lambda_j^u p_k + \lambda_j^d q_k\right)\mu_j^2
\end{aligned}
$$

$$
\gamma_2^{aljd}(\Delta(lnS_t) = \frac{\hat{N}}{\left(\mathit{\Gamma}\right)^2} - 3 \tag{4.150}
$$

where,

$$\hat{N} = 8\mu_j^2\left(\frac{\lambda_j^u p_k}{\alpha_1^2} + \frac{\lambda_j^d q_k}{\alpha_2^2}\right) + 2\mu_j^2\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right) + \mu_j^3\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right) + \mu_j^4\left(\lambda_j^u p_k + \lambda_j^d q_k\right)$$

and

$$\Gamma = \sigma_d^2\Delta t - \frac{2\lambda_j^u p_k}{\alpha_1^3} + 2\mu_j\left(\frac{\lambda_j^u p_k}{\alpha_1} + \frac{\lambda_j^d q_k}{\alpha_2}\right) + \left(\lambda_j^u p_k + \lambda_j^d q_k\right)\mu_j^2$$

4.7.3 The derivation of the moments of the MDRJD process via the LK formula

Similarly, $\mathbb{E}, \mathcal{V}, \gamma_1, \gamma_2$ for the MDRJD process was derived as:

$$\mathbb{E}^{mdrjd}(\Delta(lnS_t) = \left(\mu - \frac{1}{2\sigma^2}\right)\Delta t + \mu_j\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t$$

$$\mathcal{V}^{mdrjd}(\Delta(lnS_t) = \sigma_d^2\Delta t + \mu_j\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t$$

$$\gamma_1^{mdrjd}(\Delta(lnS_t) = \frac{\mu_j^3\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t}{\left(\sigma_d^2\Delta t + \mu_j\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t\right)^{3/2}}$$

$$\gamma_2^{mdrjd}(\Delta(lnS_t) = \frac{\mu_j^4\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t}{\left(\sigma_d^2\Delta t + \mu_j\left(\frac{p\lambda_j^u}{\sigma_j^u} + \frac{q\lambda_j^d}{\sigma_j^d}\right)\Delta t\right)^{2}} - 3$$

4.7.4 Optimal parameter estimation in the stock price models

In the sequel, the Maximum Likelihood Estimation (MLE) method was employed to obtain the optimal parameters in the models, given the likelihood function of their respective probability density functions given as:

$$L(x_i; \theta) = \prod_{i=1}^{n} g_{\Delta(lnS_t)}(x_i) \tag{4.151}$$

Maximizing the log of equation (4.151) is equivalent to minimizing the negative log-likelihood function given as:

$$-lnL(x_i; \theta) = -\sum_{i=1}^{n} lng_{\Delta(lnS_t)}(x_i) \tag{4.152}$$

Thus, the models' optimal values were computed using equation (4.152) in the R-CODES.

4.7.5 Results of parameter estimation in the GBM model

In order to fit the empirical stock indices data obtained from the three stock markets, into the GBM model, it is important to determine the initial and optimal values of the drift and volatlity parameters (μ_d and σ^2) associated with the GBM model. Hence, using equations (4.136) and (4.137) the initial values for μ_d and σ were obtained. Also, equation (4.152) shall be employed to obtain the optimal parameters. The initial and optimal estimates of μ_d and σ were given respectively in Tables 4.4 and 4.5 below. The above mentioned were obtained via the Rcodes and henceforth the **initial estimates** were represented with *int* and **optimal estimates** were represented with *opm*.

Table 4.4: Estimated initial parameters in the GBM model for the Stock indices.

$\hat{\theta}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{int}$	0.083	0.136	0.026
$\hat{\sigma}_d^{int}$	0.169	0.219	0.169

The Table 4.4 reports the initial estimates of the parameters in the geometric Brownian model described in chapter three of this thesis. These include the drift and volatility respectively, $\hat{\mu}$ and $\hat{\sigma}_d$. The initial values: $\hat{\mu}_d^{int}$ and $\hat{\sigma}_d^{int}$ for the three stock markets were obtained using equations (4.136) and (4.137). In the Table 4.4, concerning $\hat{\sigma}_d^{int}$, the results show that the diffusive process possesses a higher randomness in the UK stock market than the Nigerian and Japan markets.

Table 4.5: Estimated optimal parameters in the GBM model for the Stock indices

$\hat{\theta}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{opm}$	0.083	0.147	0.035
$\hat{\sigma}_d^{opm}$	0.169	0.219	0.169

The Table 4.5 reports the optimal values of $\hat{\mu}$ and $\hat{\sigma}_d$ for the three stock markets. The values are obtained by inputting the initial estimates reported in Table 4.4 above into the GBM log-returns density in equation (3.95) and then, via equation (4.152). The values of $\hat{\sigma}_d^{opm}$ were found to be the same as the values of $\hat{\sigma}_d^{int}$.

4.7.6 Comparison of the densities of the modelled GBM with stock indices

Next, the optimal values of μ_d and σ^2 were used to plot the probability density function of the GBM-modelled log returns, compared with the empirical densities of the NASI, UKSMI, and the JSMI log returns. The Figures 4.10 - 4.12, gives the graphs of the densities of the modelled GBM and empirical log returns of the three stock markets. The peakedness of the modelled GBM density was found to be very different from that of the empirical densities especially in the Nigerian case. However, good fits of the tails of the modelled GBM with regards to the empirical were observed.

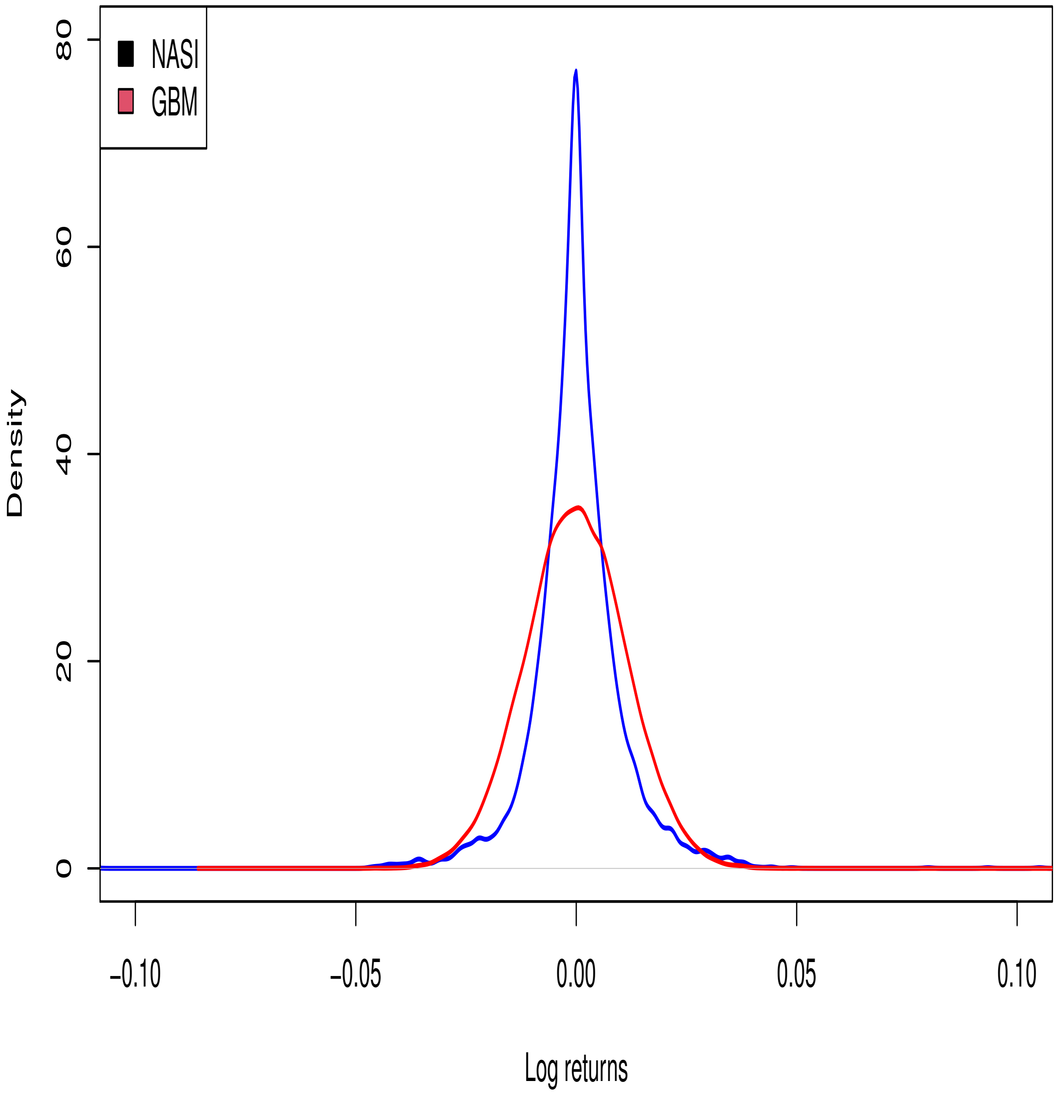

Figure 4.10: Graphs of the densities of modelled GBM and empirical log returns of the NASI

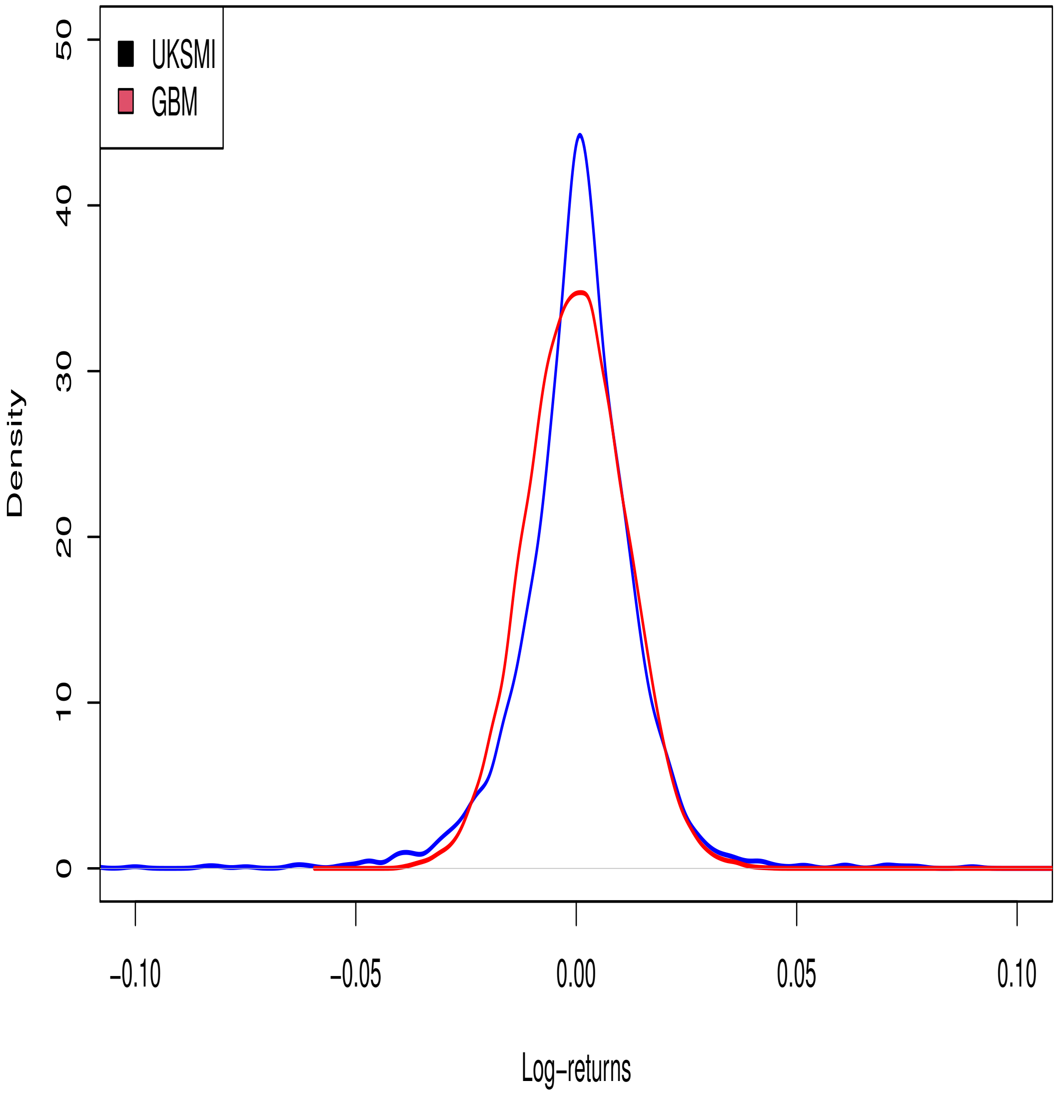

Figure 4.11: Graphs of the densities of modelled GBM and empirical log returns of the UKSMI

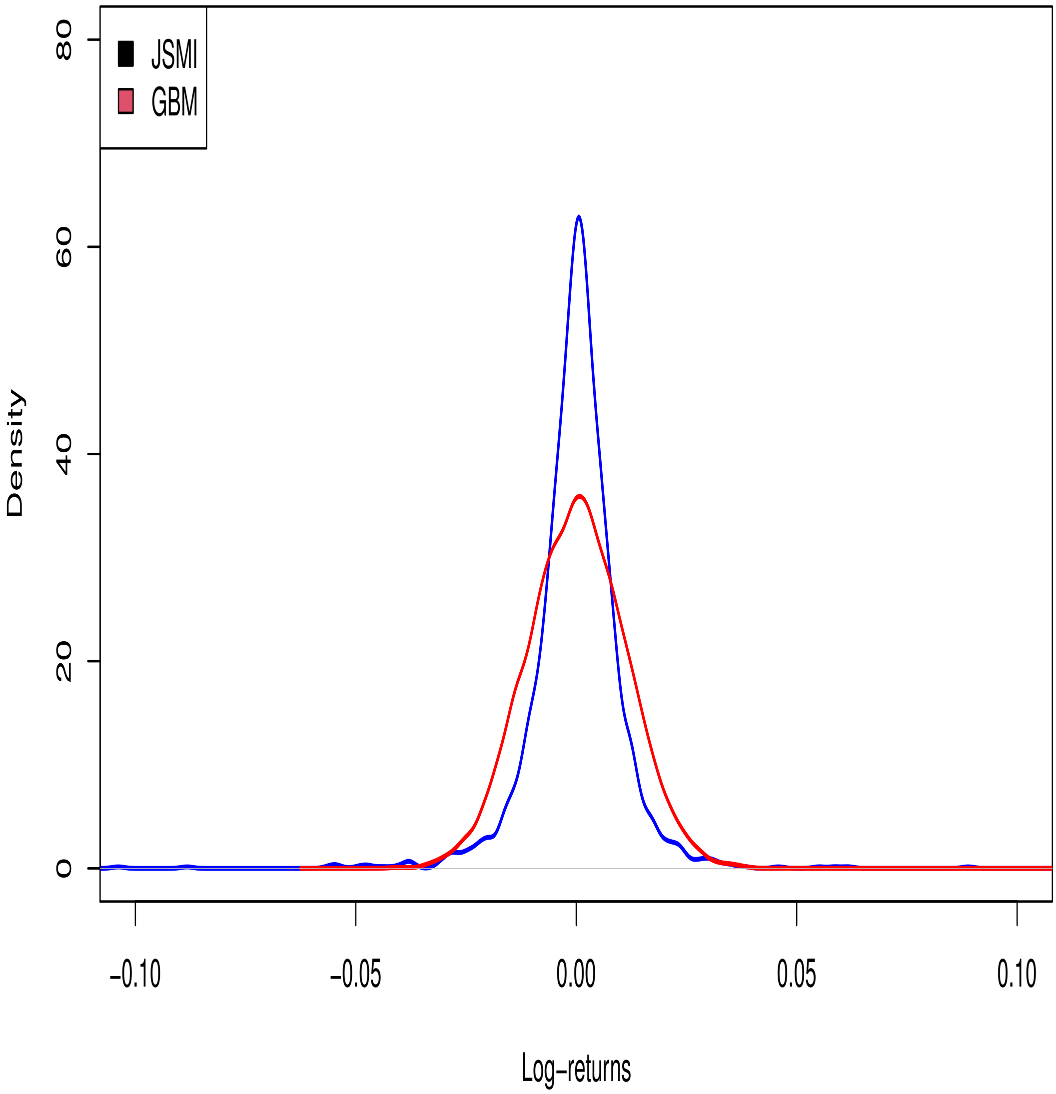

Figure 4.12: Graphs of the densities of modelled GBM and empirical log returns of the JSMI

4.7.7 Results of parameter estimation in the NJD model

In the symmetric NJD model, the initial values of the five parameters, respectively : $\mu_d, \sigma, \mu_j \sigma_j$ and λ for the empirical NASI, UKSMI, and JSMI data were found. A threshold of jumps $\epsilon = 0.02$ was assumed for the empirical log returns, such that it was assumed that a jump occur if $|X_{\Delta,t}| \geq \epsilon$. Then, the optimal parameters in the model via the method in equation (4.152), given the density of the model in equation (3.104). The results obtained were presented in Tables 4.6 and 4.7.

Table 4.6: Initial parameters in the symmetric NJD model for the stock indices

$\hat{\theta}^{int}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{int}$	0.033	0.089	0.228
$\hat{\sigma}_d^{int}$	0.111	0.112	0.136
$\hat{\mu}_j^{int}$	0.003	-0.005	-0.004
$\hat{\sigma}_j^{int}$	0.032	0.035	0.033
$\hat{\lambda}^{int}$	16.40	13.00	26.25

The Table 4.6 reports the initial estimates of the parameters in the NJD described in equation (3.103). These include: $\hat{\mu}_d$, $\hat{\sigma}_d$, $\hat{\mu}_j$, $\hat{\sigma}$, and $\hat{\lambda}$, respectively, diffusive drift, volatility, mean and volatility of the jump size, and the jump intensity. These values were obtained for the three stock markets using equations (4.131), (4.136), (4.137), (4.142) and (4.143).

Table 4.7: Optimal parameters in the symmetric NJD model for the Stock indices

$\hat{\theta}^{opm}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{opm}$	-0.047	0.136	0.320
$\hat{\sigma}_d^{opm}$	0.074	0.097	0.148
$\hat{\mu}_j^{opm}$	0.001	-0.002	-0.005
$\hat{\sigma}_j^{opm}$	0.013	0.017	0.024
$\hat{\lambda}^{opm}$	122.1	54.2	43.4

The Table 4.7 gives the optimal parameter estimates for the NJD model via the maximum likelihood estimation method in equation (4.152). The values of $\hat{\mu}_j^{opm}$ (the mean of the jump size) obtained for the three stock markets depict more upward jumps in the Nigerian case and more downward jumps in the UK and Japan markets. Based on the values of $\hat{\lambda}^{opm}$, the jump intensity was found to be higher in the Nigerian market, showing that there were more jumps in the price process of the NASI price process than the UKSMI and JSMI.

4.7.8 Comparison of the densities of the modelled NJD with stock indices

The optimal parameters obtained in the tables above were fitted into the density function of the symmetric NJD-modelled log returns and compared with the empirical density functions of the NASI, UKSMI and JSMI log returns. The graphs of the densities were given in Figures 4.13, 4.14 and 4.15 below. It was obvious that the peakedness in the densities of the modelled NJD model was found to be better than the GBM modelled densities in terms of its fitness with the empirical densities.

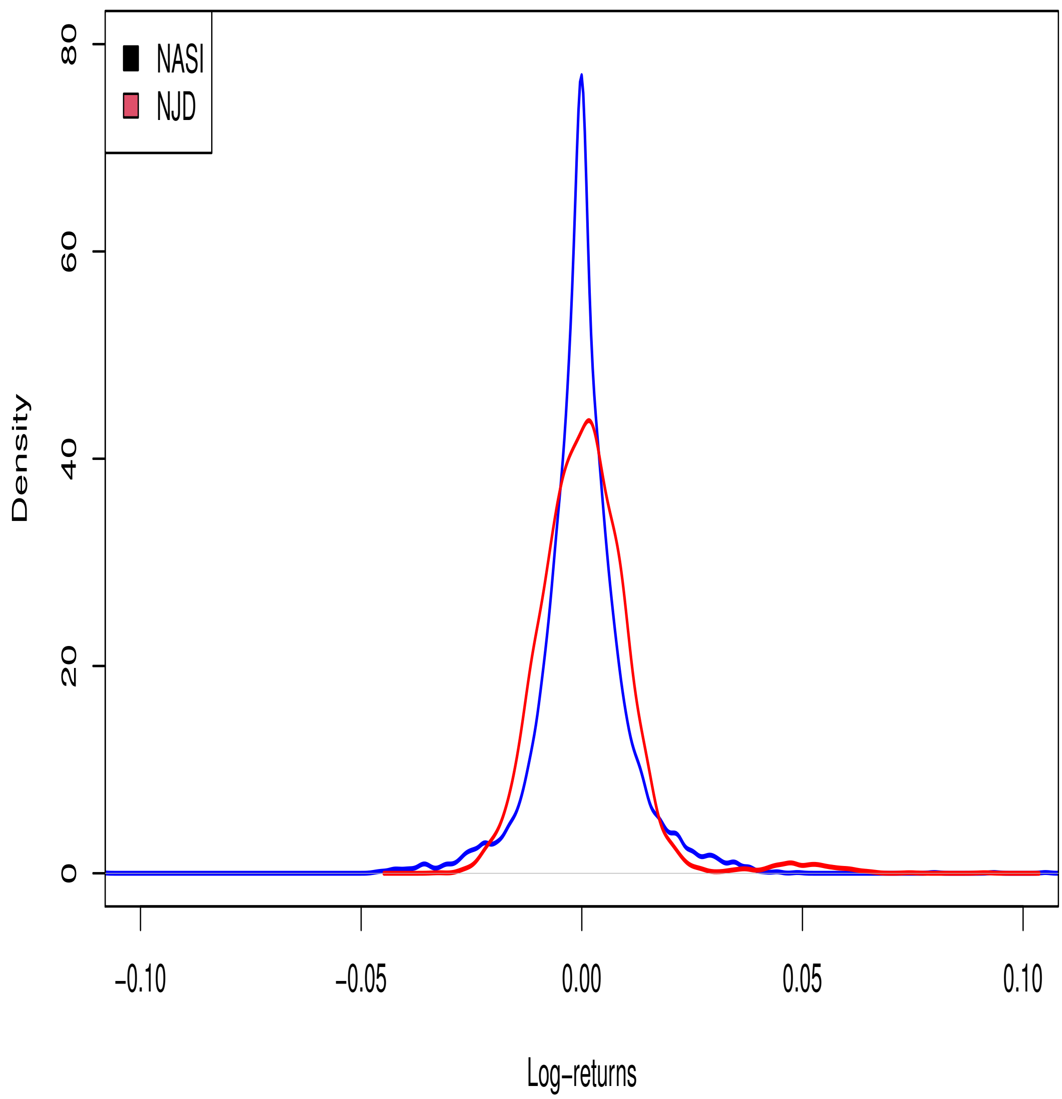

Figure 4.13: Graphs of the densities of modelled NJD and empirical log returns of the NASI

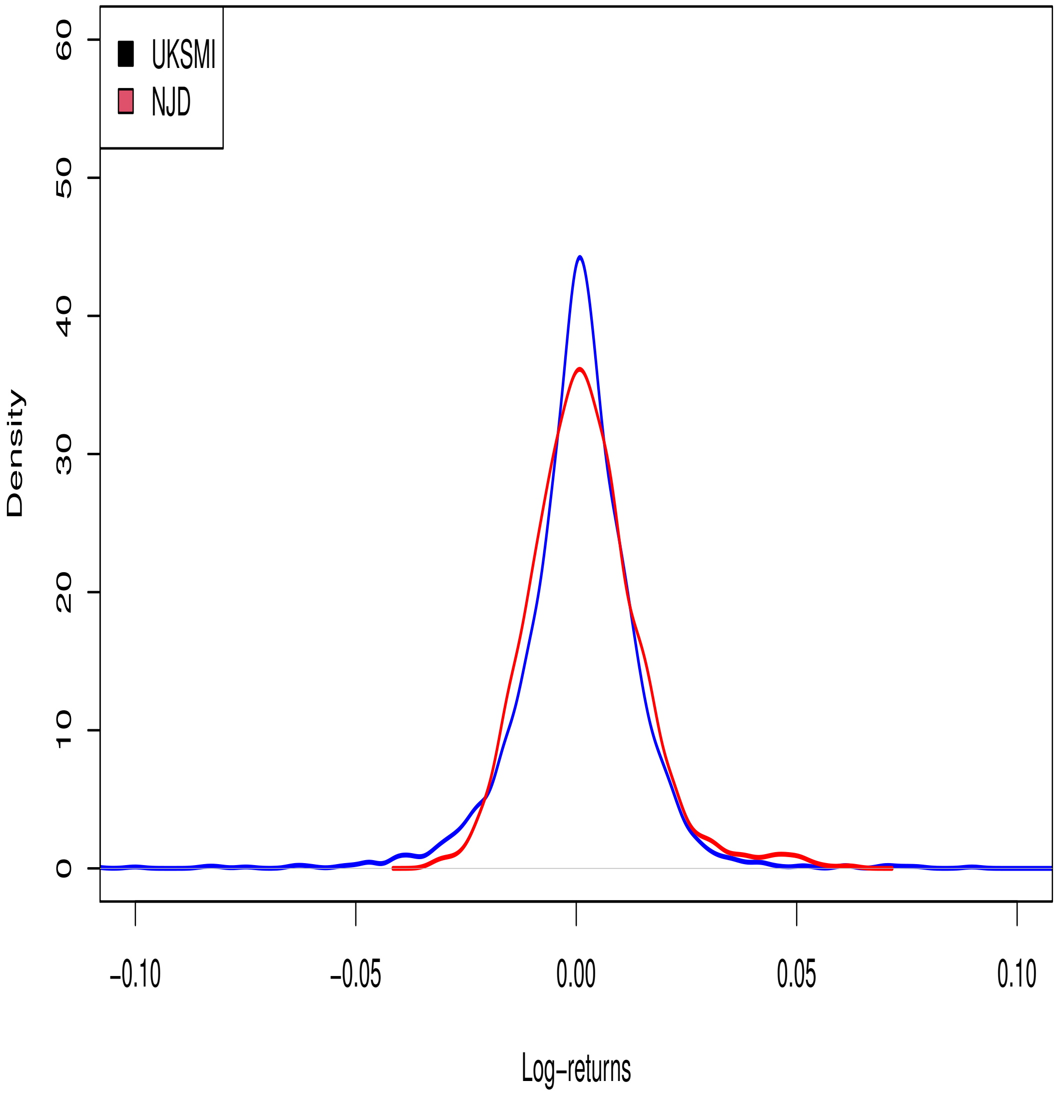

Figure 4.14: Graphs of the densities of modelled NJD and empirical log returns of the UKSMI

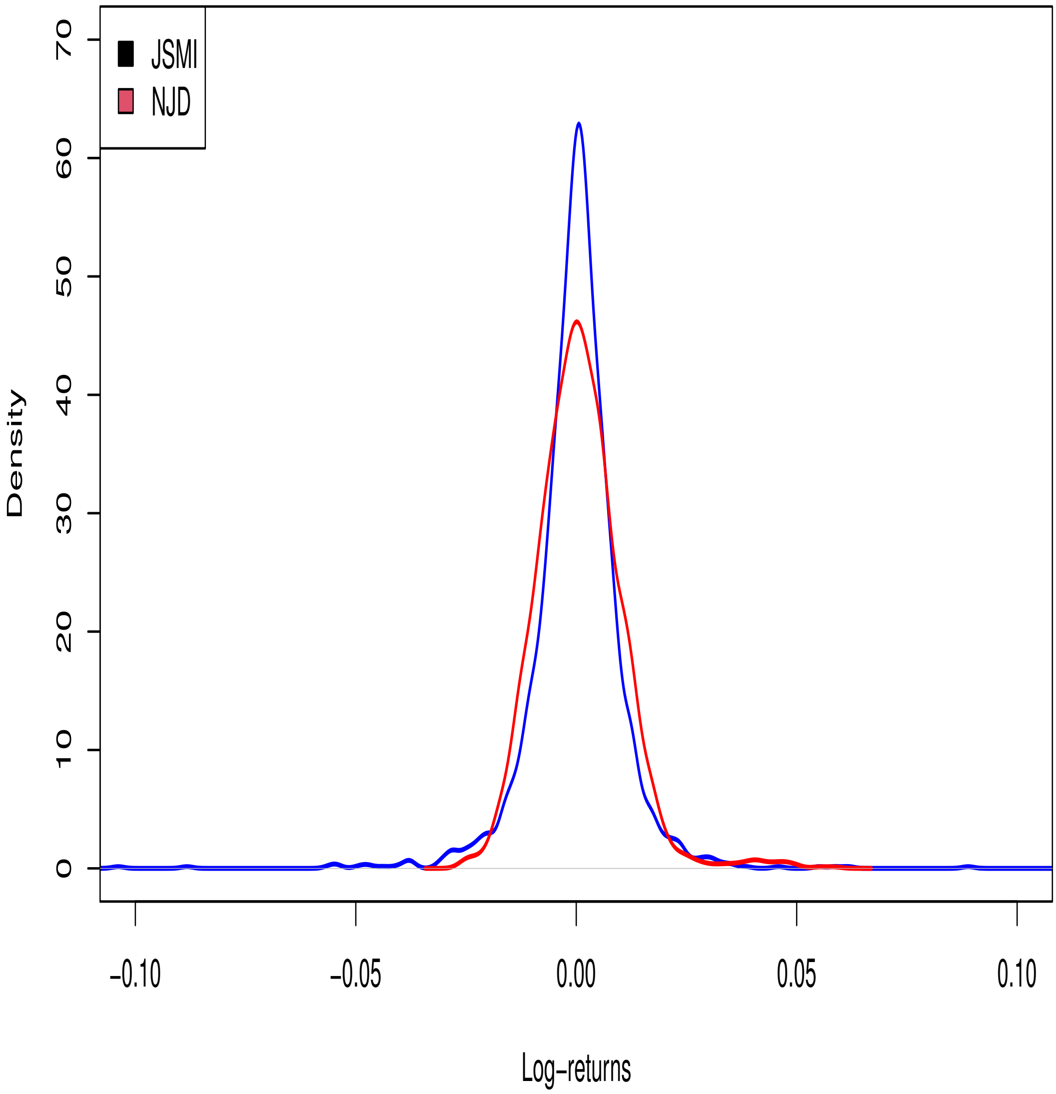

Figure 4.15: Graphs of the densities of modelled NJD and empirical log returns of the JSMI

4.7.9 Results of parameter estimation in the DEJD model

In a similar manner, the initial parameters in the asymmetric DEJD model as described in the previous section for $\mu_d, \sigma, \eta_1, \eta_2, p, q$ and λ for the empirical NASI, UKSMI, and JSMI data were obtained. Under the family of the skewed jump-diffusion model, it was assumed that the jump process is splitted into two, namely, the upward jump process $(X^u_{\Delta,t})$ and the downward jump-process $(X^d_{\Delta,t})$, such that an upward jump was said to have occurred if $X_{\Delta,t} > \epsilon$, and a downward jump was said to have occurred if $X_{\Delta,t} < -\epsilon$. Here, we assume that the threshold of jumps $\epsilon = 0.02$. Then, we shall also obtain the optimal parameters in the model via the method in equation (4.152), given the density of the DEJD model in equation (3.106). The results obtained were presented in Tables 4.8 and 4.9.

Table 4.8: Initial parameters in the asymmetric DEJD model for the Stock indices

$\hat{\theta}^{int}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{int}$	0.0343	0.0993	0.2388
$\hat{\sigma}^{int}$	0.1070	0.1086	0.1522
$\hat{\lambda}^{int}$	18.618	14.945	15.6741
$\hat{p}^{int}$	0.5406	0.035	0.4186
$\hat{q}^{int}$	0.4593	13.00	0.5814
$\hat{\eta}_1^{int.}$	35.8056	35.2546	26.9886
$\hat{\eta}_2^{int}$	35.7536	32.0274	26.4789

Table 4.8 above gives the initial values of the parameters: $\mu_d, \sigma_d \lambda p, q, \eta_1, \eta_2$ in the double exponential jump-diffusion model using the the three stock market data via equations (4.131), (4.136), (4.137), (4.144) and (4.145). The jump intensities were found to be higher under the DEJD model than the NJD model.

Table 4.9: Optimal parameters in the asymmetric DEJD model for the Stock indices

$\hat{\theta}^{opm}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{opm}$	0.0323	0.0993	0.2388
$\hat{\sigma}^{opm}$	0.1123	0.1086	0.1522
$\hat{\lambda}^{opm}$	16.618	26.2449	13.6741
$\hat{p}^{opm}$	0.5476	0.4676	0.4486
$\hat{q}^{opm}$	0.4524	0.5324	0.5514
$\hat{\eta}_1^{opm}$	34.5112	33.3438	33.9321
$\hat{\eta}_2^{opm}$	34.171	30.9421	30.1072

The results of the optimal parameters in the DEJD model were presented in Table 4.9 above. The values obtained for $\hat{\lambda}^{opm}$ showed that the number of jumps are more in the UKSMI market than the NASI and JSMI. However, the values of $\hat{p}^{opm}$ and $\hat{q}^{opm}$ obtained show that the upward jump frequency was higher in NASI and downward jump frequency was higher in the UKSMI and JSMI markets. Recall from equation (4.145), that the mean upward and downward jump sizes were respectively, $\hat{\eta}_1 = (\mathbb{E}(Q_j^u))^{-1}$ and $\hat{\eta}_2 = (\mathbb{E}(Q_j^d))^{-1}$. The values obtained for $\hat{\eta}_1$ and $\hat{\eta}_2$ indicate bigger downward jumps in the Nigerian market than the UK and Japan markets.

4.7.10 Comparison of the densities of the modelled DEJD with stock indices

Next, the density function of the asymmetric DEJD-modelled log returns and the empirical densities of the NASI, UKSMI, and the JSMI log returns were compared, based on the estimated optimal parameters obtained in Table 4.9 above. The graphs of the densities were given in Figures 4.16, 4.17 and 4.18 below. The peakedness of the densities of the DEJD modelled log returns was seen to be better than that of the GBM and NJD models.

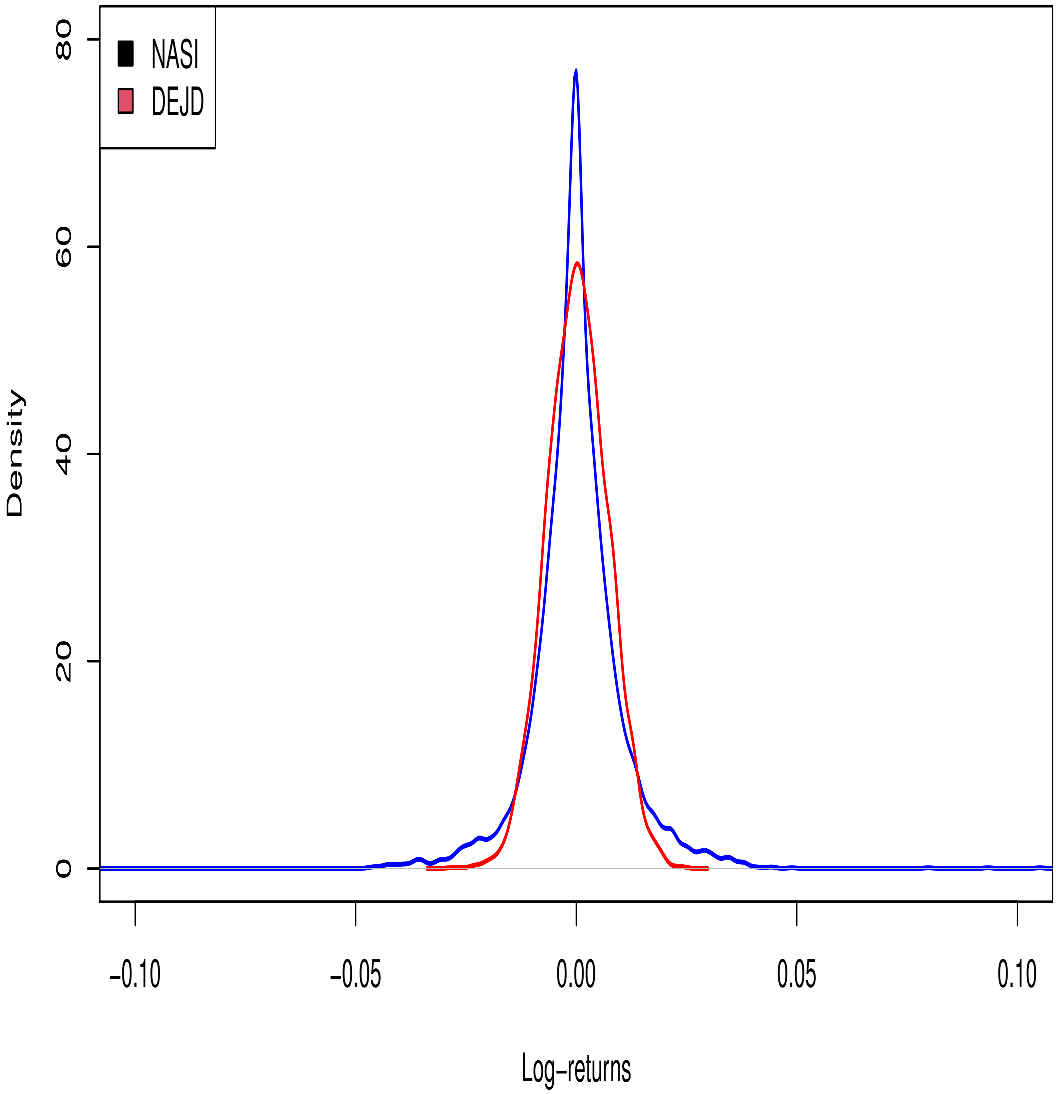

Figure 4.16: Graphs of the densities of modelled DEJD and empirical log returns of the NASI

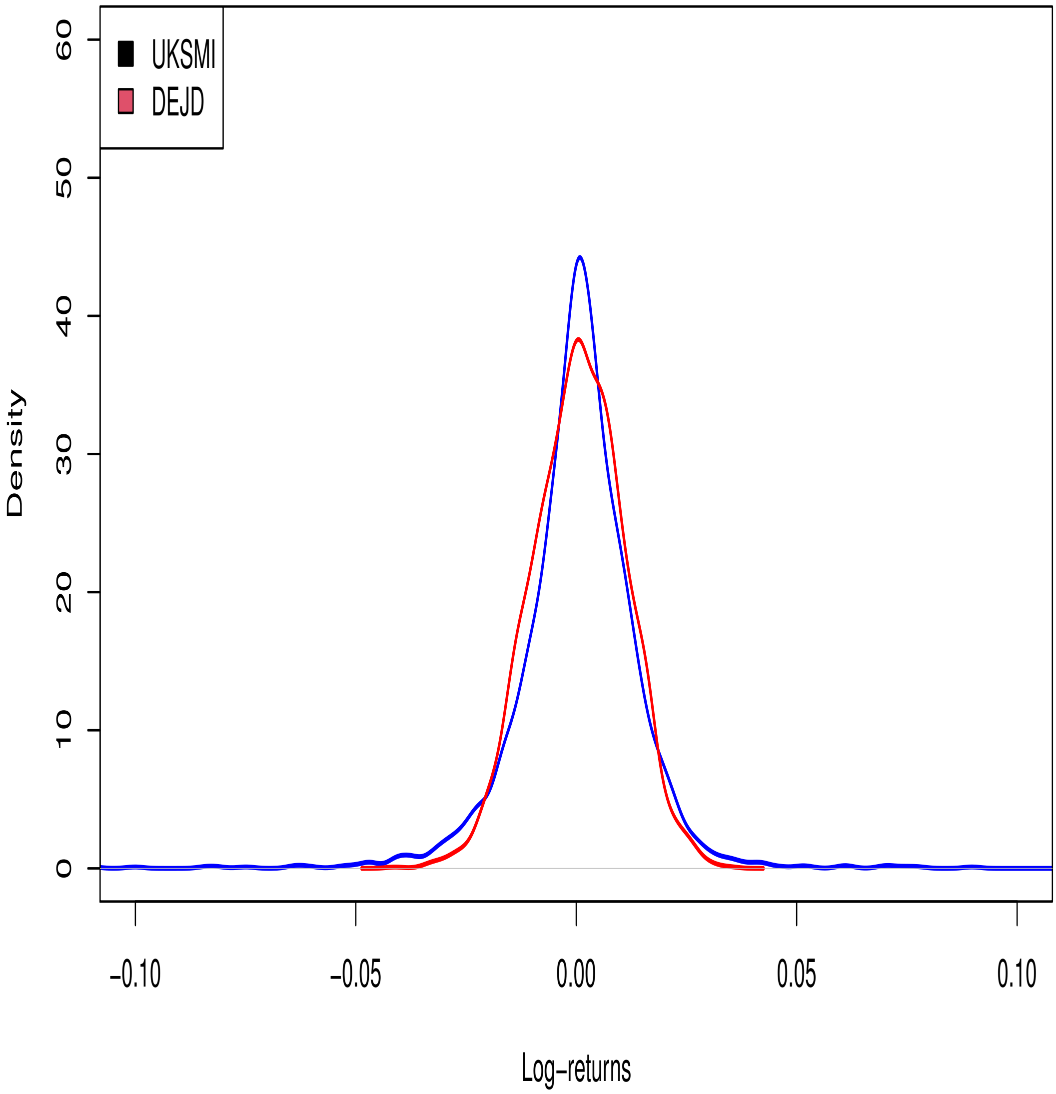

Figure 4.17: Graphs of the densities of modelled DEJD and empirical log returns of the UKSMI

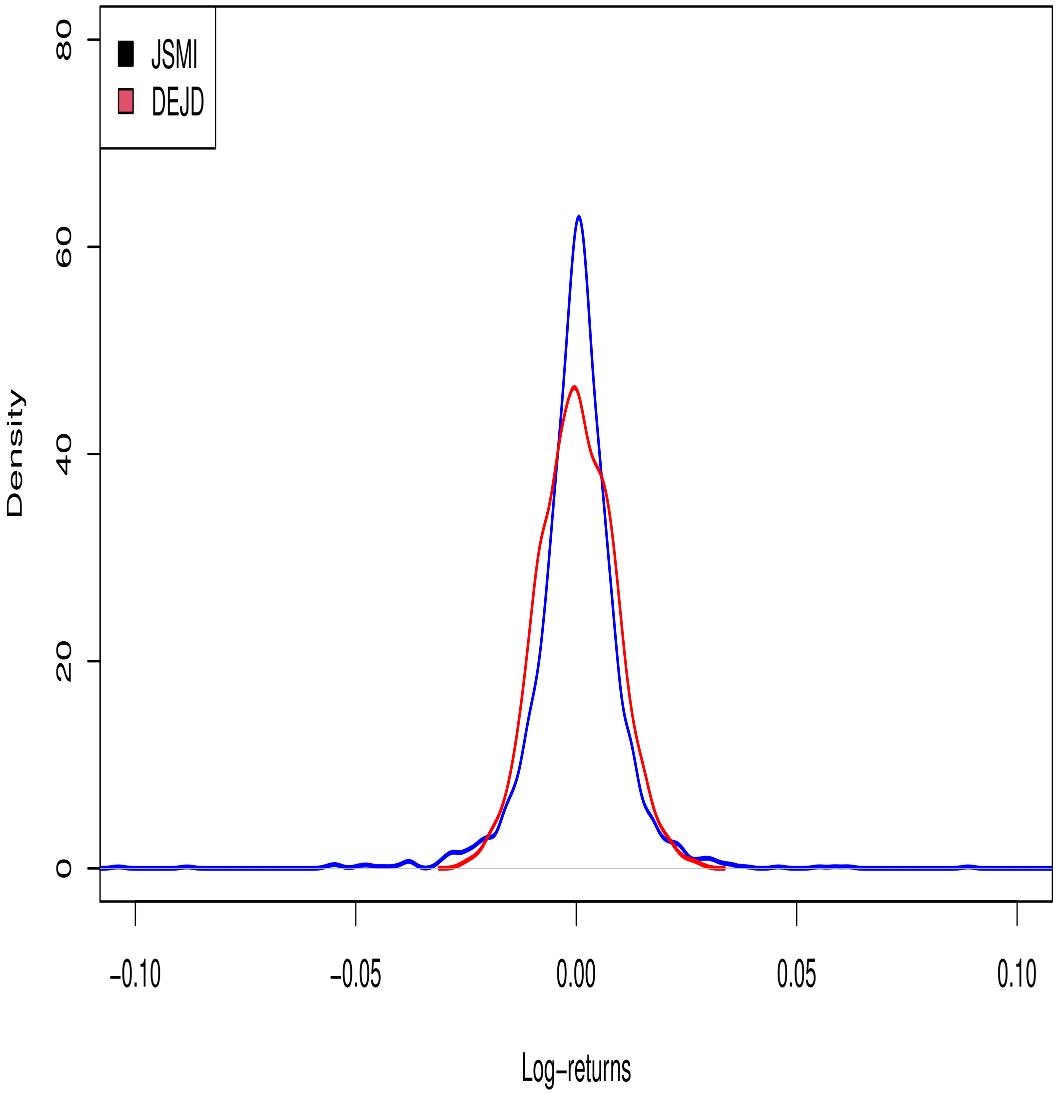

Figure 4.18: Graphs of the densities of modelled DEJD and empirical log returns of the JSMI

4.7.11 Results of parameter estimation in the ALJD model

To obtain the initial parameters' estimates in the Asymmetric Laplace jump-diffusion (ALJD) model with parameters : $\mu_d, \sigma, \mu_j, p_\kappa, q_\kappa \;\; \alpha_1, \alpha_2$ and λ, it followed from equation (4.75) that the density of $Q_j \sim AL^*(\mu_j, \sigma^2, \kappa)$ is given as:

$$\begin{aligned} f_{Q_t}(x) &= p_\kappa \alpha_1 exp(-\alpha_1(x-\mu_j))\mathbf{1}_{[\mu_j,\infty)}(x) \\ &+ q_\kappa \alpha_2 \exp(\alpha_2(x-\mu_j))\mathbf{1}_{(-\infty,\mu_j)}(x) \end{aligned} \tag{4.153}$$

In the above, p_κ and q_κ are the tail probabilities assigned to each sides of μ_j. Hence,

$$q_\kappa = \mathbb{P}(Q_j > \mu_j) = 1 - \Phi(Q_j \leq \mu_j) = 1 - \Phi(\mu_j) \tag{4.154}$$

and,

$$p_\kappa = 1 - q_\kappa = \frac{1}{1+\kappa^2} \tag{4.155}$$

Also, $\frac{1}{\alpha_1}$ and $\frac{1}{\alpha_2}$ are respectively the means of Q_j^u and Q_j^d respectively. Thus

$$\alpha_1 = \left(\mathbb{E}(Q_j^u)\right)^{-1} = \frac{(\kappa\sqrt{2})}{\sigma}; \qquad \alpha_2 = \left(\mathbb{E}(Q_j^d)\right)^{-1} = \left|\frac{\sqrt{2}}{\sigma\kappa}\right| \tag{4.156}$$

Similarly, the estimates of $\hat{\lambda}, \hat{\mu}_d, \hat{\sigma}$ and $\hat{\mu}_j$ can be obtained respectively from equations (4.131), (4.136) and (4.137) above. Owing to the same conditions for $Q^u_{\Delta,t}$ and $Q^d_{\Delta,t}$, the threshold of jumps ϵ was taken as 0.02. Using the above descriptions of the parameters, the initial estimates for the stock market indices were obtained and presented in Tables 4.10 and 4.11 below.

$\hat{\theta}^{int}$	NASI	UKSMI	JSMI
$\hat{\mu}_d$	0.0335	0.2277	0.0890
$\hat{\sigma}$	0.1106	0.1363	0.1118
$\hat{\alpha}_1$	34.5112	33.3438	33.9321
$\hat{\alpha}_2$	34.1371	30.9421	30.1072
$\hat{\lambda}^u$	8.9781	8.2719	5.8322
$\hat{\lambda}^d$	7.4187	7.9730	7.1688
$\hat{p}_\kappa$	0.5476	0.4676	0.4486
$\hat{q}_\kappa$	0.4524	0.5324	0.5514
$\hat{\mu}_j$	0.0001	0.0002	0.0002

Table 4.10: Estimated initial parameters in the ALJD model for the Stock indices

In Table 4.10, the details of the initial parameters in the ALJD model: $\hat{\mu}, \hat{\sigma}, \hat{\alpha}_1, \hat{\alpha}_2$ $\hat{\lambda}^u, \hat{\lambda}^d, \hat{p}_\kappa, \hat{q}_\kappa, \hat{\mu}_j$ were presented. These values were obtained using equations (4.154, 4.155, 4.154 4.132, 4.133) with the stock markets data. Here, the jump intensities $\hat{\lambda}^u_j$ and $\hat{\lambda}^d_j$ were assumed to be independent and a new parameter (μ_j) that differs from the existing ones in literature was introduced.

Table 4.11: Optimal parameters in the ALJD model for the Stock indices

$\hat{\theta}^{opm}$	NASI	UKSMI	JSMI
$\hat{\mu}_d$	0.0335	0.3715	0.8262
$\hat{\sigma}$	0.1453	0.0278	0.1124
$\hat{\alpha}_1$	29.6117	26.8358	38.2628
$\hat{\alpha}_2$	35.5559	33.2650	31.8505
$\hat{\lambda}^u$	8.6894	8.7694	5.8874
$\hat{\lambda}^d$	7.0137	7.5479	7.3417
$\hat{p}_\kappa$	0.5476	0.5674	0.4412
$\hat{q}_\kappa$	0.4524	0.4326	0.5888
$\hat{\mu}_j$	0.7024	0.7771	0.4269

Based on the MLE method and subject to the density function of the ALJD model in equation (4.65), the optimal values were also obtained for the parameters and presented in Table 4.11 above. The results showed a higher drift in the JSMI and higher volatility in NASI. The upward jump sizes are found to be higher in the NASI and UKSMI. Similarly, the jump intensities were found to be higher in NASI and UKSMI stock markets. The non zero parameter μ_j obtained, differed in a great sense from its initial estimates. The optimal values obtained for μ_j showed a positive shift in the distribution of the jump process.

4.7.12 Comparison of the densities of the modelled ALJD with stock indices

In the sequel, the density function of the ALJD-modelled log returns and the empirical NASI, UKSMI, and the JSMI log returns was obtained, based on the estimated optimal parameters obtained in Table 4.11 above. The graphs of the densities were given in Figures 4.19, 4.20 and 4.21 below.

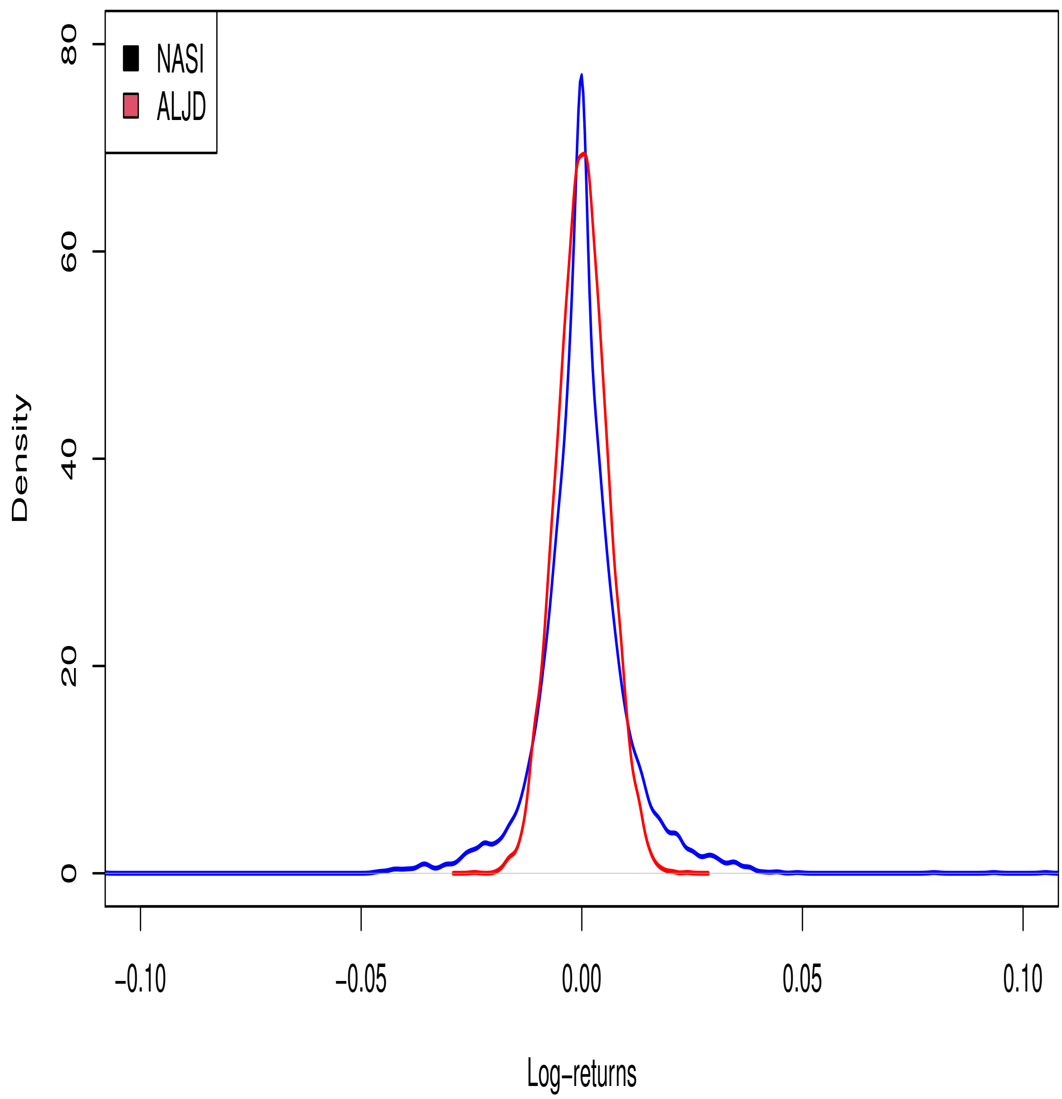

Figure 4.19: Graphs of the densities of modelled ALJD and empirical log returns of the NASI

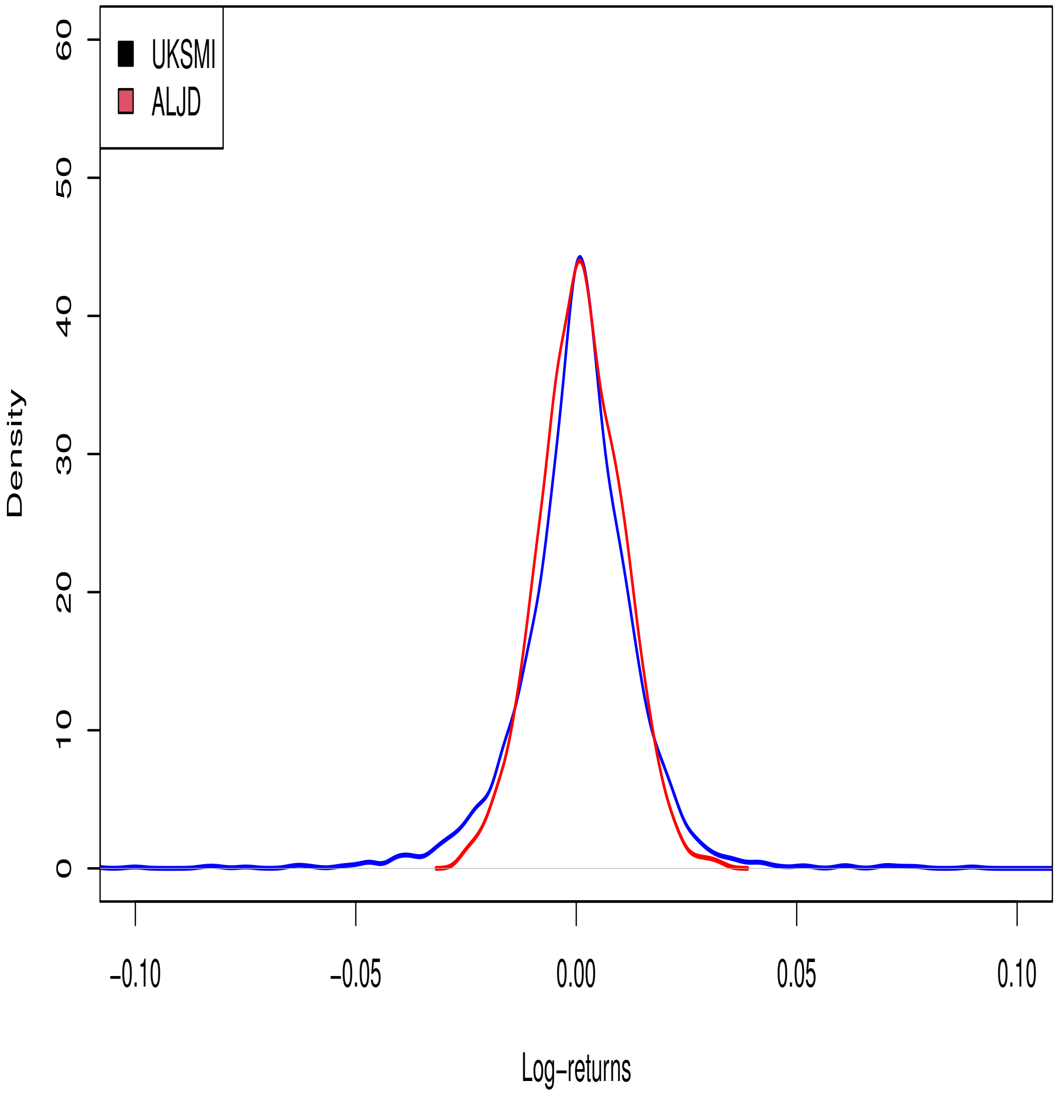

Figure 4.20: Graphs of the densities of modelled ALJD and empirical log returns of the UKSMI

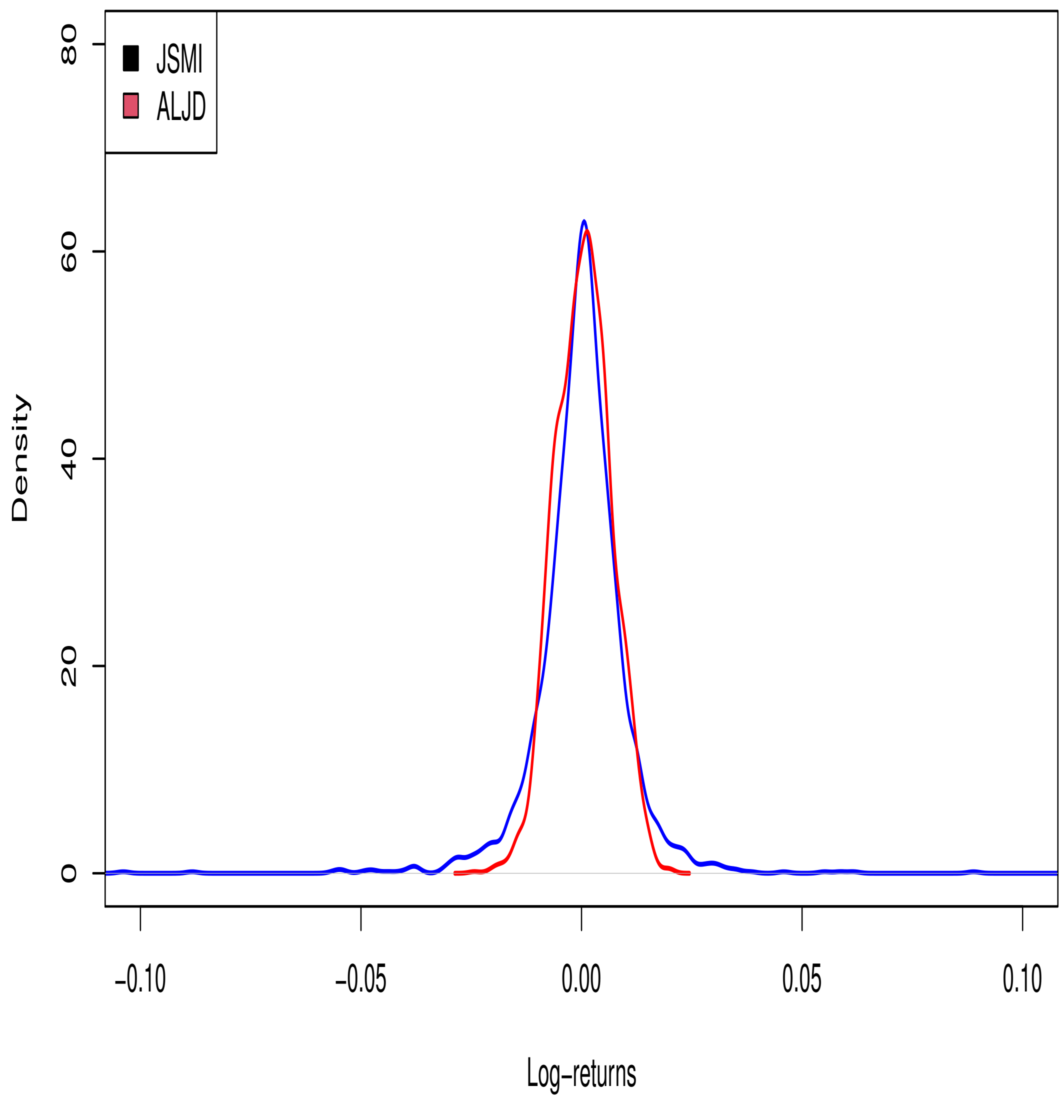

Figure 4.21: Graphs of the densities of modelled ALJD and empirical log returns of the JSMI

4.7.13 Results of parameter estimation in the MDRJD model

The results of the initial and optimal parameters in the modified double Rayleigh jump-diffusion model were obtained and presented below. The initial estimates of the parameters: $\mu_d, \sigma, \sigma_j^u, \sigma_j^d, \lambda_j^u, \lambda_j^d, p, q$ were obtained empirically using equations (4.133), (4.134), (4.135), (4.136), (4.139) and (4.140) for the empirical NASI, UKSMI, and JSMI data. In the analysis, an upward jump was said to have occurred if $X_{\Delta,t} > \epsilon$, (under the assumption that the threshold of jumps was $\epsilon = 0.02$) and a downward jump was said to have occurred if $X_{\Delta,t} < -\epsilon$. Then, the optimal parameters in the model via the method in equation (4.152), given the density of the MDR jump-diffusion model in equation (4.89) were obtained. The results were presented in Tables 4.12 and 4.13.

Table 4.12: Initial parameters in the MDRJD model for the Stock indices

$\hat{\theta}^{int}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{int}$	0.034	0.228	0.089
$\hat{\sigma}^{int}$	0.111	0.136	0.112
$\hat{\sigma}_j^u$	0.023	0.024	0.024
$\hat{\sigma}_j^d$	0.023	0.026	0.027
$\hat{\lambda}_j^u$	8.9797	12.2779	5.8350
$\hat{\lambda}_j^d$	7.4201	13.9797	7.1722
$\hat{p}^{int}$	0.548	0.468	0.449
$\hat{q}^{int}$	0.452	0.532	0.551
$\hat{\mu}_j^{int}$	0.000	0.000	0.000

Table 4.12 gives the estimated initial values of the parameters in the MDRJD model via equations (4.136), (4.137), (4.144), (4.145), (4.132) and (4.133) with the stock market indices data.

Table 4.13: Optimal parameters in the MDRJD model for the stock indices

$\hat{\theta}^{opm}$	NASI	UKSMI	JSMI
$\hat{\mu}_d^{opm}$	0.0562	0.2317	0.1235
$\hat{\sigma}^{opm}$	0.0860	0.1226	0.0765
$\hat{\sigma}_j^u$	0.0054	0.0147	0.0074
$\hat{\sigma}_j^d$	0.0519	0.0370	0.0566
$\hat{\lambda}_j^u$	9.4197	12.3929	5.8251
$\hat{\lambda}_j^d$	7.4499	14.1039	7.1932
$\hat{p}^{opm}$	0.5571	0.4872	0.4494
$\hat{q}^{opm}$	0.4443	0.5324	0.5506
$\hat{\mu}_j^{opm}$	0.0412	0.0842	0.0329

Table 4.13 gives the optimal values of the MDRJD model in the NASI, UKSMI and JSMI markets. The volatility, upward and downward jump intensities under the UKSMI market were found to be higher. However, the jump intensity for the up jumps was found to be higher than for the down jumps. The above results also showed that the number of up jumps were found to be more in the Nigerian market than the UK and Japan markets. A positive shift in the jump process was also observed for the MDRJD model. And finally it was observed that the density of $J(Q_j^u)$ peaked at 0.0054, 0.0147 and 0.0074 respectively in the NASI, UKSMI and JSMI; and the density of $J(Q_j^d)$ peaked at 0.0519, 0.0370 and 0.0566. Notably, the parameters: $\hat{\sigma}_j^u$ and $\hat{\sigma}_j^d$ evince this.

4.7.14 Comparison of the densities of the modelled MDRJD with stock indices

Here, the densities of the MDRJD-modelled log returns were compared with the empirical NASI, UKSMI, and the JSMI log returns, using the estimated optimal parameters obtained in Table 4.13 above. The graphs of the densities were given in Figures 4.22, 4.23 and 4.24 below.

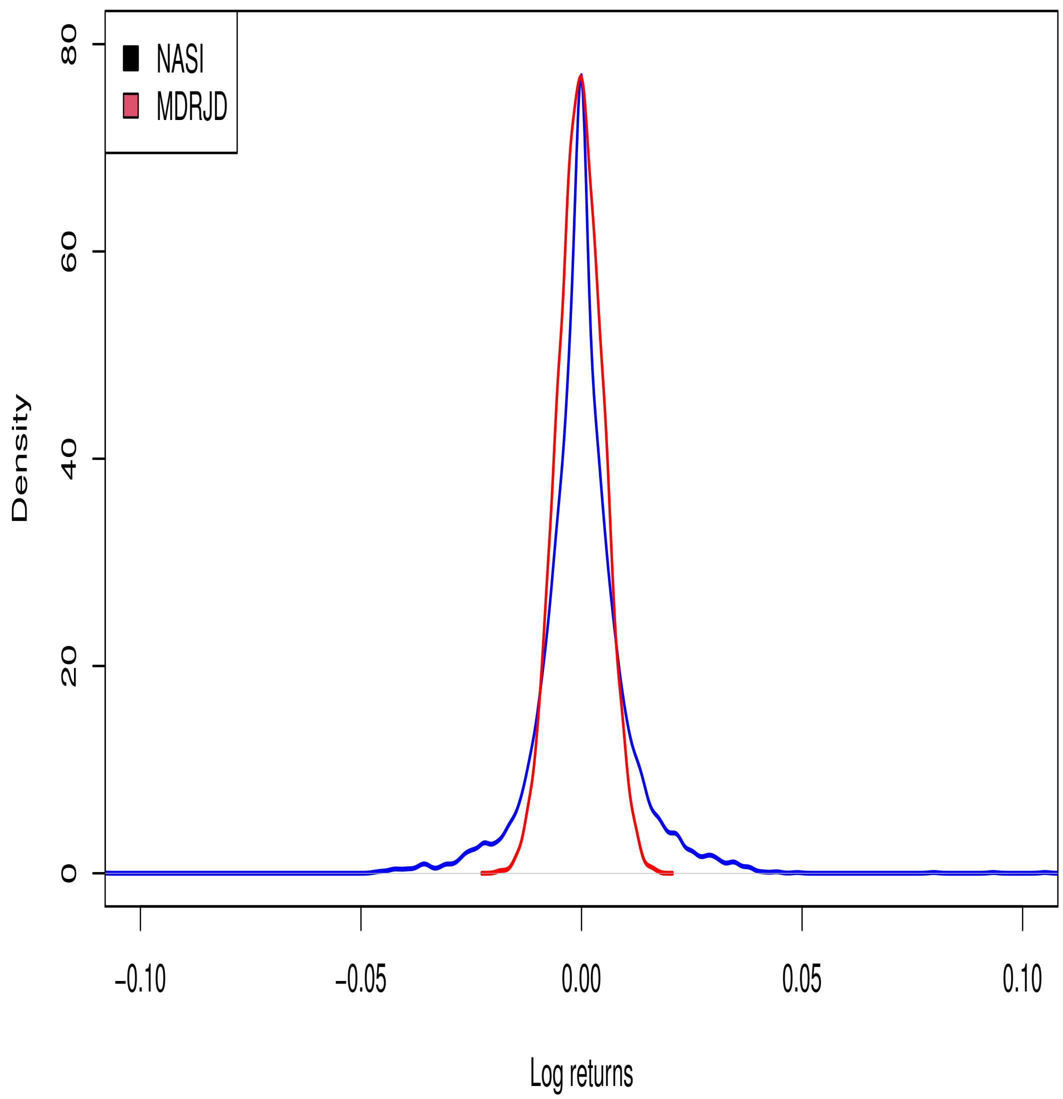

Figure 4.22: Graphs of the densities of modelled MDRJD and empirical log returns of the NASI

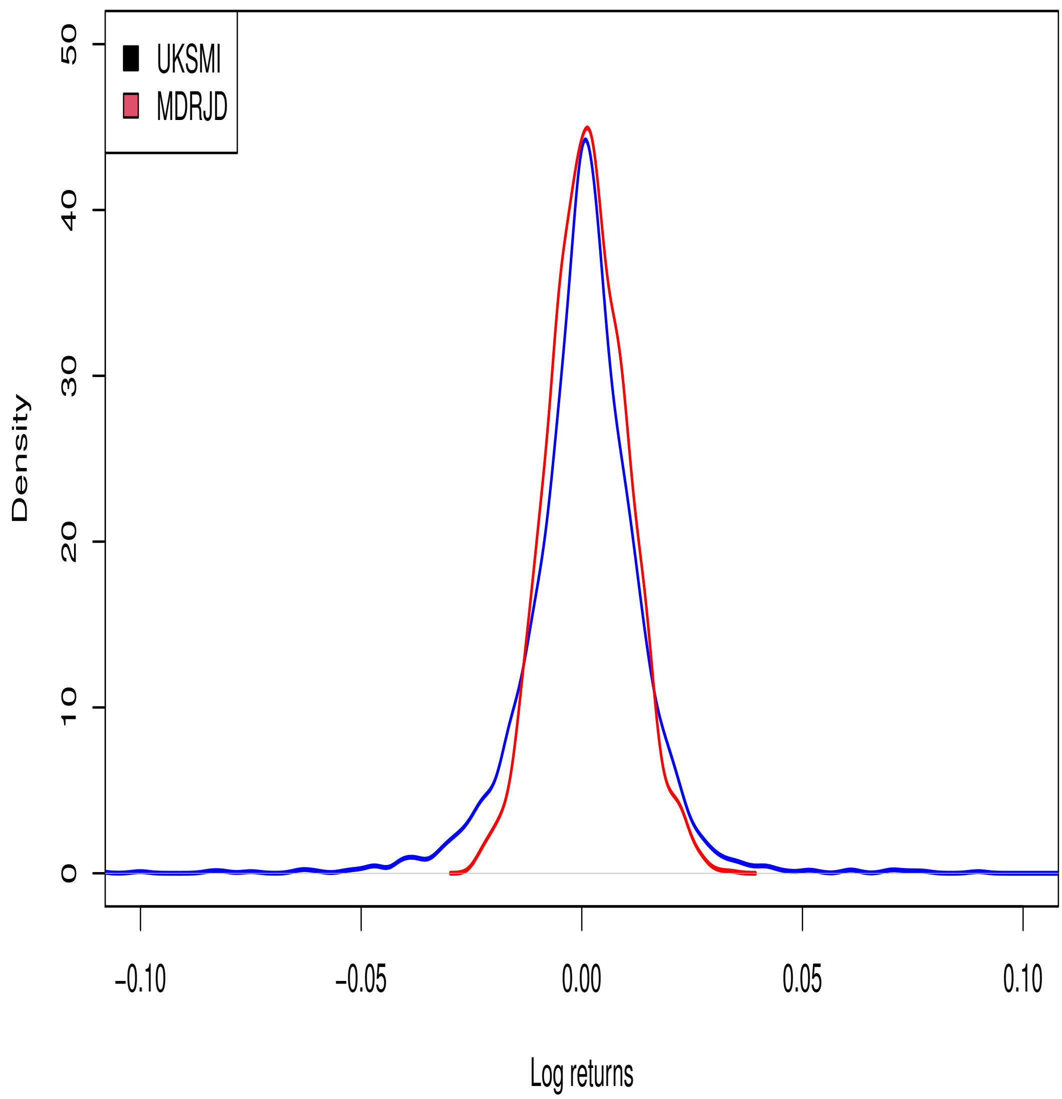

Figure 4.23: Graph of the denisities of the modelled MDRJD and empirical log returns of the UKSMI

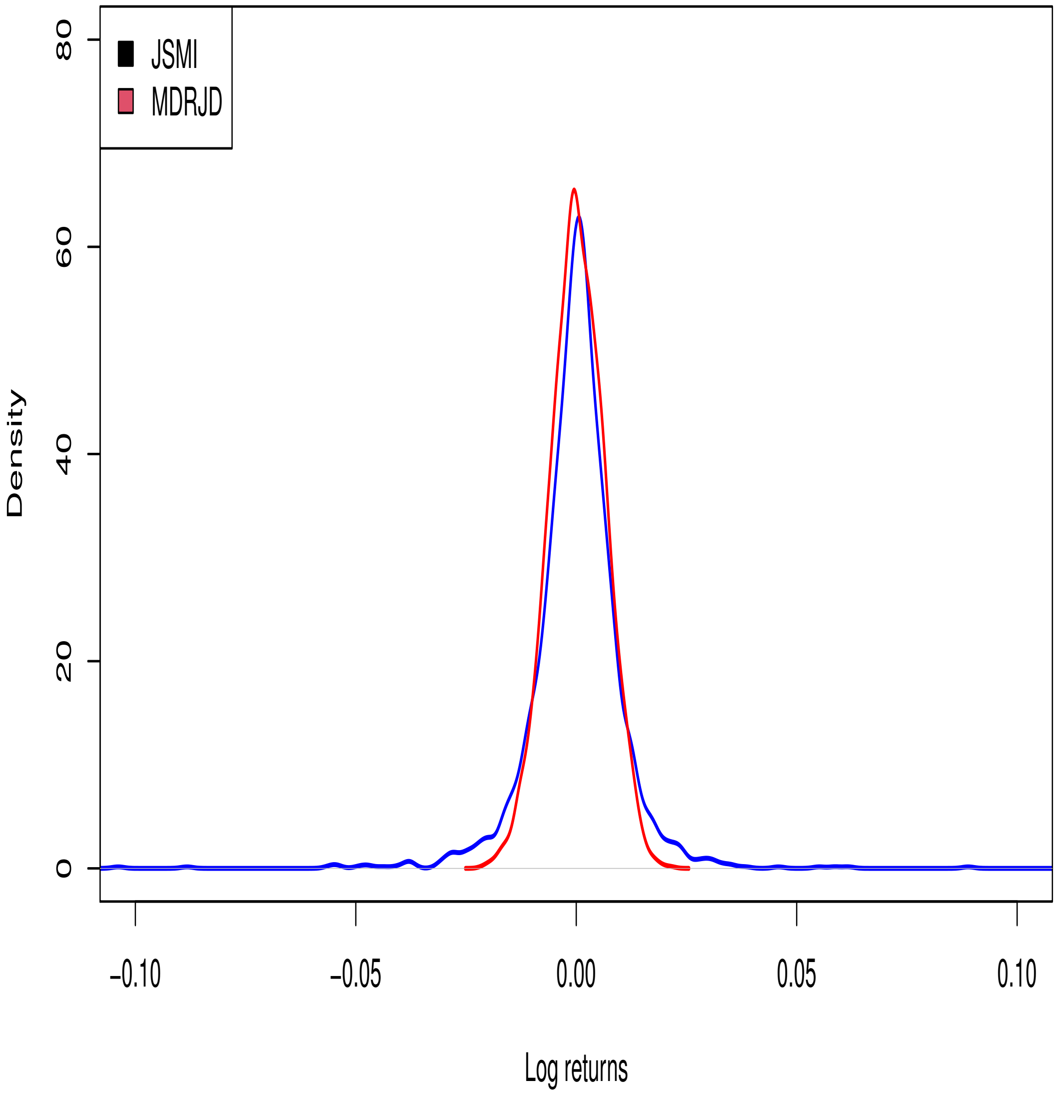

Figure 4.24: Graphs of the densities of modelled MDRJD and empirical log returns of the JSMI

4.8 STUDY SEVEN

Sensitivity analysis of varied jump-threshold on the parameters in the models

Determining the threshold of jumps in the plots of the log returns is very important and its choice depends on the empirical data set under study. For example in the plots of the stock Indices log returns given in the Figure 4.25 below, by assuming that $\epsilon = max\left(\Delta(ln\tilde{S}_t)\right)$, then $n(Q_j^u) = 0$ and $n(Q_j^d) = 0$. Given that the value of ϵ, is taken to be as small as $min\left(\Delta(ln\tilde{S}_t)\right)$, then, it implies that the entire process becomes a jump-process, which is not true (based on the evidences in Figures 4.7-4.9) in the actual sense. The choice of the threshold of jumps is a major determinant of the output of the results obtained for the initial and optimal parameters in the models. Therefore, a sensitivity analysis of the varied threshold of jumps on the parameters of the jump-diffusion models was carried out.

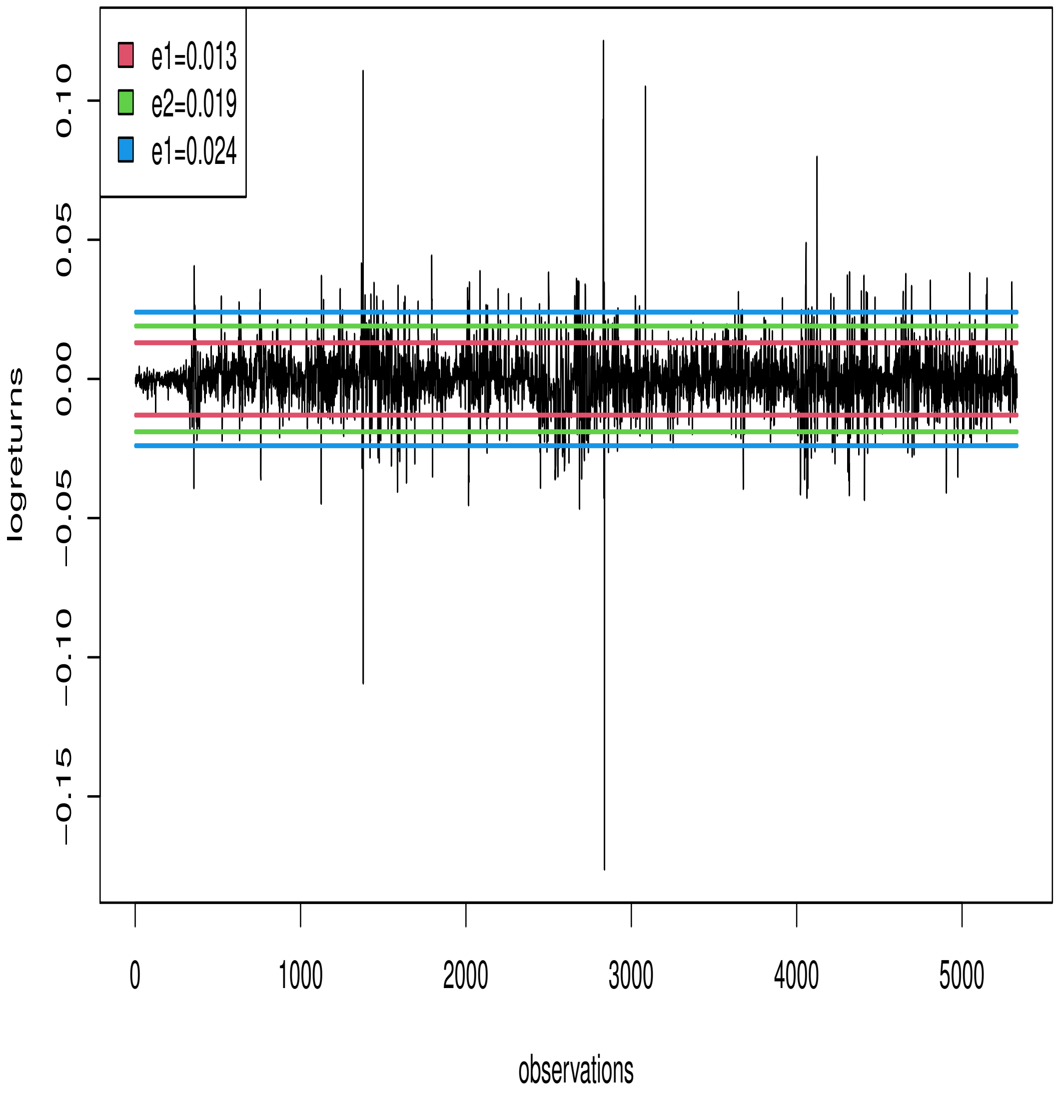

Figure 4.25: Plot showing the varied threshold of jumps in the log returns

4.8.1 Sensitivity Analysis of varied jump-threshold on the parameters in the NJD model

Here, a sensitivity analysis of the threshold of jumps under the symmetric NJD model for the NASI, UKSMI and the JSMI data was carried out. Assume that $\epsilon \in (0.01, 0.025)$, specifically, five threshold of jumps, that is, $\epsilon_1 = 0.013, \epsilon_2 = 0.016, \epsilon_3 = 0.019, \epsilon_4 = 0.021$ and $\epsilon_5 = 0.023$. The Tables 4.14, 4.15 and 4.16 below report the results of the sensitivity analysis for the different threshold of jumps, where the initial and optimal parameters were represented as $\hat{\theta}^{int}$ and $\hat{\theta}^{opm}$, respectively.

Table 4.14: Sensitivity analysis of ϵ on the parameters in the symmetric NJD model for the Nigerian Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	0.07	-0.05	0.07	-0.05	0.08	-0.05	0.07	-0.05	0.02	-0.05
$\hat{\sigma}_d$	0.16	0.07	0.16	0.07	0.15	0.07	0.14	0.07	0.11	0.07
$\hat{\lambda}$	0.33	122.1	0.33	122.1	0.76	122.1	3.83	122.1	14.32	122.1
$\hat{\mu}_j$	0.03	0.00	0.03	0.00	0.01	0.00	0.00	0.00	0.00	0.00
$\hat{\sigma}_j$	0.12	0.01	0.12	0.01	0.09	0.01	0.05	0.01	0.03	0.01

Table 4.14 shows the sensitivity of the parameters in the NJD model for the Nigerian market to the varied threshold of jumps. The different values of epsilon $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and ϵ_5 depict five jump threshold. The results obtained show that the optimal parameters were not sensitive to the varied threshold of jumps since they remain the same under different threshold.

Table 4.15: Sensitivity analysis of ϵ on the parameters in the symmetric NJD model for the UK Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	0.238	-0.04	0.232	-0.04	0.211	-0.04	0.199	-0.04	0.176	-0.04
$\hat{\sigma}_d$	0.126	0.02	0.154	0.02	0.167	0.02	0.174	0.02	0.183	0.02
$\hat{\lambda}$	0.265	26.93	0.702	26.95	0.748	26.93	0.94	26.93	0.594	26.93
$\hat{\mu}_j$	- 0.001	3.50	-0.007	3.50	-0.010	3.50	-0.012	3.50	-0.011	3.50
$\hat{\sigma}_j$	0.033	2.05	0.040	2.05	0.048	2.05	0.054	2.05	0.46	2.05

The Table 4.15 presents the sensitivity of the parameters in the NJD model for the UK stock market to the varied threshold of jumps. The results obtained show that the optimal parameters were not sensitive to the varied threshold of jumps since they remain the same under different threshold.

Table 4.16: Sensitivity analysis of ϵ on the parameters in the symmetric NJD model for the Japan Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	0.095	0.296	0.082	0.296	0.097	0.296	0.094	0.296	0.092	0.296
$\hat{\sigma}_d$	0.090	0.786	0.101	0.786	0.109	0.786	0.114	0.786	0.122	0.786
$\hat{\lambda}$	0.077	36.514	0.263	36.514	0.581	36.514	0.543	36.514	0.776	36.514
$\hat{\mu}_j$	-0.002	2.096	-0.003	2.096	-0.005	2.096	- 0.007	2.096	-0.009	2.096
$\hat{\sigma}_j$	0.025	0.105	0.029	0.105	0.033	0.105	0.036	0.105	0.041	0.105

The Table 4.16 reports the sensitivity of the parameters in the NJD model for the Japan stock market to the varied threshold of jumps. The results obtained showed that the optimal parameters were not sensitive to the varied threshold of jumps since they remain the same under different jump threshold.

4.8.2 Sensitivity analysis of varied jump-threshold on the parameters in the DEJD model

A sensitivity analysis of the threshold of jumps under the asymmetric DEJD model for the NASI, UKSMI and the JSMI data was carried out. Assume that $\epsilon_+ \in (0.01, 0.025)$, and $\epsilon_- \in (-0.025, -0.01)$, specifically, five threshold of jumps, that is, $\epsilon_1 = 0.013, \epsilon_2 = 0.016, \epsilon_3 = 0.019, \epsilon_4 = 0.021$ and $\epsilon_5 = 0.023$. Tables 4.17-4.19 below report the results of the sensitivity analysis for the different threshold of jumps, where the initial and optimal parameters were represented as $\hat{\theta}^{int}$ and $\hat{\theta}^{opm}$, respectively.

Table 4.17: Sensitivity analysis of ϵ on the parameters in the asymmetric DEJD model for the Nigerian Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	-0.012	0.112	0.010	0.213	0.033	0.263	0.045	0.224	0.054	0.472
$\hat{\sigma}_d$	0.087	0.912	0.098	0.184	0.108	0.019	0.113	0.142	0.122	0.217
$\hat{\eta}_1$	45.816	46.324	40.056	39.998	35.468	36.468	33.229	32.124	29.817	29.351
$\hat{\eta}_2$	44.923	45.923	39.352	40.352	35.288	35.867	33.383	31.879	30.148	31.152
$\hat{\lambda}$	35.439	35.346	25.139	24.897	18.003	18.642	14.827	13.964	10.396	9.877
$\hat{p}$	0.5557	0.5280	0.5545	0.5732	0.5430	0.5620	0.5350	0.578	0.532	0.601
$\hat{q}$	0.4443	0.4720	0.4455	0.4268	0.4570	0.4380	0.4650	0.422	0.468	0.399

Table 4.17 above gives the sensitivity of the parameters in the DEJD model for the Nigerian market to the varied threshold of jumps. The different values of epsilon $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and ϵ_5 depict five jump threshold. The results obtained showed that the optimal parameters were quite sensitive to the varied threshold of jumps since they change with the varied threshold.

Table 4.18: Sensitivity analysis of ϵ on the parameters in the asymmetric DEJD model for the UK Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	0.159	0.067	0.197	0.897	0.209	0.087	0.242	0.092	0.239	0.213
$\hat{\sigma}_d$	0.101	0.099	0.119	0.214	0.131	0.112	0.139	0.143	0.151	0.102
$\hat{\eta}_1$	46.925	46.462	39.925	38.461	35.293	36.157	31.973	31.973	27.496	26.596
$\hat{\eta}_2$	41.846	40.982	36.216	36.216	31.963	34.386	30.334	31.426	26.853	25.342
$\hat{\lambda}$	61.117	61.024	41.555	40.486	29.647	28.896	24.058	25.061	16.403	16.997
$\hat{p}$	0.527	0.564	0.503	0.552	0.488	0.443	0.450	0.450	0.422	0.422
$\hat{q}$	0.473	0.436	0.497	0.448	0.512	0.557	0.551	0.551	0.578	0.578

Table 4.18 shows the sensitivity of the parameters in the DEJD model for the UK stock market to the varied threshold of jumps. The different values of epsilon: $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and ϵ_5 depict five jump thresholds. The results obtained showed that the optimal parameters were quite sensitive to the varied threshold of jumps since they change with the varied threshold.

Table 4.19: Sensitivity analysis of ϵ on the parameters in the asymmetric DEJD model for the Japan Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_j$	0.095	0.084	0.082	0.818	0.097	0.099	0.094	0.086	0.092	-0.086
$\hat{\sigma}_d$	0.689	0.569	0.101	0.099	0.109	0.016	0.114	0.126	0.122	0.120
$\hat{\eta}_1$	47.622	46.864	41.012	41.022	35.028	35.028	32.267	32.186	27.9317	27.152
$\hat{\eta}_2$	43.114	42.340	36.023	36.346	31.655	31.355	29.055	30.121	29.315	30.231
$\hat{\lambda}$	32.077	32.198	21.263	21.599	14.580	13.348	11.541	12.012	7.776	8.001
$\hat{p}$	0.477	0.467	0.480	0.480	0.442	0.442	0.436	0.432	0.406	0.419
$\hat{q}$	0.523	0.533	0.520	0.520	0.558	0.558	0.569	0.568	0.594	0.591

Table 4.19 the sensitivity of the parameters in the DEJD model for the Japan stock market to the varied threshold of jumps. The different values of ϵ: $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ and ϵ_5 depict five jump thresholds. The results obtained showed that the optimal parameters were also quite sensitive to the varied threshold of jumps since they change with the varied threshold.

4.8.3 Sensitivity analysis of varied jump-threshold on the parameters in the ALJD model

The sensitivity analysis of the threshold of jumps under the Asymmetric Laplace JD model, is very important since for the first time, the intensities of the jumps in the process were viewed under two processes. This analysis was geared towards detecting the effect of the varied threshold of jumps on the parameters associated with the intensity of jumps. Assume that $\epsilon_+ \in (0.01, 0.025)$, and $\epsilon_- \in (-0.025, -0.01)$, specifically, five threshold of jumps, that is, $\epsilon_1 = 0.013, \epsilon_2 = 0.016, \epsilon_3 = 0.019, \epsilon_4 = 0.021$ and $\epsilon_5 = 0.023$. Tables 4.20, 4.21 and 4.22 below report the results of the sensitivity analysis for the different threshold of jumps, where the initial and optimal parameters were represented as $\hat{\theta}^{int}$ and $\hat{\theta}^{opm}$, respectively.

Table 4.20: Sensitivity analysis of ϵ on the parameters in the asymmetric Laplace JD model for the Nigerian Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	-0.012	0.661	0.010	0.351	0.033	0.716	0.045	0.591	0.054	0.959
$\hat{\sigma}_d$	0.087	0.800	0.098	0.269	0.108	0.342	0.113	0.381	0.122	0.413
$\hat{\alpha}_1$	45.816	43.368	40.056	38.904	35.468	36.459	33.229	35.945	29.817	22.177
$\hat{\alpha}_2$	44.922	42.096	39.352	35.610	35.288	30.549	33.383	25.928	30.148	34.046
$\hat{\lambda}^u$	19.75	13.94	9.78	7.35	6.27	6.98	5.23	5.91	5.21	4.98
$\hat{\lambda}^d$	15.69	11.19	8.64	6.37	5.45	5.67	5.11	4.28	4.72	4.01
$\hat{p}$	0.557	0.689	0.555	0.705	0.543	0.584	0.535	0.564	0.532	0.553
$\hat{q}$	0.443	0.311	0.446	0.294	0.457	0.416	0.465	0.437	0.468	0.447
$\hat{\mu}_j$	0.0001	0.328	0.0001	0.439	0.0001	0.331	0.0001	0.575	0.0001	0.381

Table 4.20 the sensitivity of the parameters in the asymmetric ALJD model for the Nigerian stock market to the varied threshold of jumps. The results obtained showed that the initial and optimal parameters were also quite sensitive to the varied threshold of jumps since they change with the varied threshold, with the exception of the parameter μ_j in the case of the initial.

Table 4.21: Sensitivity analysis of ϵ on the parameters in the asymmetric Laplace JD model for the UK Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	0.160	0.704	0.197	0.690	0.209	0.917	0.242	0.793	0.239	0.267
$\hat{\sigma}_d$	0.101	0.088	0.119	0.655	0.132	0.993	0.139	0.806	0.151	0.245
$\hat{\alpha}_1$	46.925	45.554	39.325	36.675	35.223	33.509	31.972	32.862	27.496	29.176
$\hat{\alpha}_2$	41.846	41.775	36.214	35.494	31.963	28.755	30.334	31.862	26.853	27.643
$\hat{\lambda}^u$	15.25	12.94	10.78	10.35	8.16	7.47	7.23	6.19	6.21	5.98
$\hat{\lambda}^d$	13.19	12.43	9.64	7.37	6.45	6.67	6.11	5.28	4.72	4.01
$\hat{p}$	0.527	0.647	0.503	0.513	0.488	0.419	0.450	0.400	0.422	0.417
$\hat{q}$	0.473	0.353	0.497	0.487	0.513	0.581	0.551	0.600	0.578	0.583
$\hat{\mu}_j$	-0.001	0.521	-0.001	0.277	0.001	0.623	0.002	0.961	0.0002	0.589

Table 4.21 gives the sensitivity of the parameters in the asymmetric ALJD model for the UK stock market to the varied threshold of jumps. It could be seen from the Table that the initial and optimal parameters were quite sensitive to the varied threshold of jumps since they change with the varied threshold. The jump intensities were found to be on the decrease as ϵ increases.

Table 4.22: Sensitivity analysis of ϵ on the parameters in the asymmetric Laplace JD model for the Japan Stock Market

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	0.095	0.453	0.082	0.075	0.097	0.004	0.092	0.308	0.092	0.218
$\hat{\sigma}_d$	0.090	0.599	0.101	0.959	0.109	0.727	0.114	0.486	0.122	0.714
$\hat{\alpha}_1$	47.622	44.271	41.012	37.662	35.029	35.232	32.265	31.393	27.932	30.211
$\hat{\alpha}_2$	43.114	41.969	36.023	35.884	31.656	27.834	29.055	25.6393	25.315	18.600
$\hat{\lambda}^u$	15.75	13.94	14.87	12.11.	12.27	11.98	11.23	10.91	10.21	11.98
$\hat{\lambda}^d$	18.69	17.19	18.64	18.37	17.45	16.67	15.11	14.28	14.72	14.01
$\hat{p}$	0.477	0.382	0.480	0.434	0.442	0.458	0.432	0.432	0.406	0.389
$\hat{q}$	0.523	0.617	0.520	0.566	0.558	0.542	0.568	0.568	0.594	0.611
$\hat{\mu}_j$	0.0002	0.851	0.0003	0.8940	0.0002	0.884	0.0002	0.457	0.0002	0.347

Table 4.22 gives the sensitivity of the parameters in the asymmetric ALJD model for the Japan stock market to the varied threshold of jumps. The initial and optimal parameters were sensitive to the varied threshold of jumps since they change with the varied threshold.

4.8.4 Sensitivity analysis of varied jump-threshold on the parameters in the MDRJD model

The sensitivity analysis of the threshold of jumps under the modified double Rayleigh JD model was motivated by the standard Rayleigh distribution that peaks at the value $\epsilon_+ = \sigma_j^u$ and $\epsilon_- = \sigma_j^d$, as described by Synowiec (2008). Since a generalised form of the above described was considered, then the threshold of jumps was varied to enable us detect its sensitivity to the parameters in the modified double Rayleigh JD model. Therefore, $\epsilon_+ \in (0.01, 0.025)$, and $\epsilon_- \in (-0.025, -0.01)$ were assumed, specifically, five threshold of jumps, that is, $\epsilon_1 = 0.013, \epsilon_2 = 0.016, \epsilon_3 = 0.019, \epsilon_4 = 0.022$ and $\epsilon_5 = 0.024$. Tables 4.23, 4.24 and 4.25 below report the results of the sensitivity analysis for the different threshold of jumps, where the initial and optimal parameters were represented as $\hat{\theta}^{int}$ and $\hat{\theta}^{opm}$, respectively.

Table 4.23: Sensitivity analysis of ϵ on the parameters in the modified double Rayleigh JD model for the NASI

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	-0.012	-0.061	0.228	0.232	0.033	0.061	0.054	0.045	0.049	0.041
$\hat{\sigma}_d$	0.087	0.088	0.136	0.123	0.108	0.081	0.117	0.115	0.125	0.104
$\hat{\sigma}_j^u$	0.017	0.070	0.023	0.015	0.023	0.052	0.025	0.060	0.028	0.007
$\hat{\sigma}_j^d$	0.018	-0.003	0.026	0.037	0.023	0.052	0.025	0.060	0.001	0.064
$\hat{\lambda}^u$	18.75	17.94	19.78	18.35	16.27	16.98	15.23	15.91	14.21	14.98
$\hat{\lambda}^d$	17.69	16.19	18.64	17.37	15.45	15.67	15.11	14.28	14.72	14.01
$\hat{p}$	0.557	0.568	0.468	0.487	0.543	0.544	0.525	0.536	0.547	0.540
$\hat{q}$	0.443	0.432	0.532	0.513	0.457	0.456	0.475	0.464	0.453	0.460
$\hat{\mu}_j$	0.000	0.043	0.000	0.084	0.000	0.037	0.000	0.074	0.000	0.060

Table 4.23 gives the sensitivity of the parameters in the modified double Rayleigh JD model for the Nigerian stock market to the varied threshold of jumps. The initial and optimal parameters were sensitive to the varied threshold of jumps since they change with the varied threshold. The jump intensities values were found to decrease with larger values of ϵ

Table 4.24: Sensitivity analysis of ϵ on the parameters in the modified double Rayleigh JD model for the UKSMI

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	0.049	0.041	0.197	0.194	0.209	0.236	0.237	0.263	0.232	0.248
$\hat{\sigma}_d$	0.125	0.104	0.119	0.099	0.132	0.115	0.144	0.116	0.154	0.129
$\hat{\sigma}_j^u$	0.028	0.007	0.020	0.005	0.023	0.016	0.026	0.003	0.030	0.063
$\hat{\sigma}_j^d$	0.028	0.063	0.022	0.037	0.025	0.038	0.028	0.027	0.031	0.050
$\hat{\lambda}^u$	16.75	16.94	15.78	13.35	13.45	11.67	11.11	12.28	11.72	11.01
$\hat{\lambda}^d$	13.69	14.19	14.64	13.07	14.27	12.98	13.23	13.91	12.21	11.98
$\hat{p}$	0.547	0.540	0.503	0.522	0.482	0.482	0.444	0.467	0.422	0.403
$\hat{q}$	0.453	0.460	0.497	0.478	0.518	0.518	0.556	0.533	0.578	0.597
$\hat{\mu}_j$	0.000	0.060	0.000	0.055	0.000	0.075	0.000	0.075	0.000	0.094

Table 4.24 reports the sensitivity of the parameters in the modified double Rayleigh JD model for the UK stock market to the varied threshold of jumps. The initial and optimal parameters were sensitive to the varied threshold of jumps since they change with the varied threshold. The jump intensities values were found to decrease with larger values of ϵ.

Table 4.25: Sensitivity analysis of ϵ on the parameters in the modified double Rayleigh JD model for the JSMI

θ	$\epsilon_1 = 0.013$		$\epsilon_2 = 0.016$		$\epsilon_3 = 0.019$		$\epsilon_4 = 0.021$		$\epsilon_5 = 0.024$	
	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$	$\hat{\theta}^{int}$	$\hat{\theta}^{opm}$
$\hat{\mu}_d$	0.095	0.114	0.082	0.101	0.097	0.122	0.097	0.122	0.091	0.097
$\hat{\sigma}_d$	0.089	0.062	0.101	0.078	0.109	0.084	0.109	0.084	0.123	0.094
$\hat{\sigma}_j^u$	0.017	0.040	0.020	0.004	0.023	0.000	0.023	0.000	0.030	0.009
$\hat{\sigma}_j^d$	0.019	0.045	0.022	0.042	0.025	0.053	0.025	0.053	0.033	0.053
$\hat{\lambda}^u$	12.75	12.94	11.78	11.35	11.45	12.67	10.11	11.28	10.72	9.01
$\hat{\lambda}^d$	13.69	14.19	14.64	13.07	14.27	12.98	13.23	13.91	12.21	11.98
$\hat{p}$	0.477	0.491	0.480	0.490	0.442	0.436	0.442	0.422	0.397	0.366
$\hat{q}$	0.523	0.509	0.520	0.510	0.558	0.564	0.558	0.578	0.603	0.634
$\hat{\mu}_j$	0.000	0.022	0.000	0.034	0.000	0.040	0.000	0.040	0.000	0.050

Table 4.25 reports the sensitivity of the parameters in the modified double Rayleigh JD model for the Japan stock market to the varied threshold of jumps. The initial and optimal parameters are sensitive to the varied threshold of jumps since they change with the varied threshold. The upward jump intensity value is found to decrease with larger values of ϵ. The initial estimate of μ_j is not sensitive with the varied jump threshold.

4.9 STUDY EIGHT

Suitability analysis of the models to the empirical stock market data

One of the objectives of this research was to evaluate the suitability of the distributions of the existing and proposed stock price models to the actual distributions of the empirical stock data obtained from the markets. To achieve this, a measure of the extent to which the obtained optimal parameters in the models affect their fitness to the empirical data was carried out. The suitability analysis was carried out using three methods. The first two methods were the test statistics namely: Kolmogorov and Anderson-Darling test statistics, and the third method is by comparing the basic moments of the distributions of the models with the empirical moments.

4.9.1 Suitability analysis via KS and AD statistics

The Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) statistics were given by:

$$\hat{b}_{(n,\alpha)} = max_x\left(\left|\hat{F}_{model}(x) - \hat{F}_{emp.}(x)\right|\right) \tag{4.157}$$

and

$$\hat{G}_{(n,\alpha)} = n\int_{-\infty}^{\infty}\frac{[\hat{F}_{model}(x) - \hat{F}_{emp.}(x)]^2}{F_{model}(x)(1 - F_{model}(x))}dF_{model}(x) \tag{4.158}$$

where, $\hat{F}_{model}(x)$ and $\hat{F}_{emp.}(x)$ are respectively the models' and the empirical distributions, under a significance level of $\alpha = 0.05$. Equation (4.157) will be used to ascertain the model with a better goodness of fit to the empirical sampled data. Owing to the above, the extent to which the empirical data sets follow the distributions of the modelled GBM, NJD, asymmetric DEJD, asymmetric Laplace JD, and MDR JD models was determined. In order to achieve this, the test hypotheses are defined as:

$$H_0 : \hat{F}_{model}(x) = \hat{F}_{emp}(x); \quad H_1 : \hat{F}_{model}(x) \neq \hat{F}_{emp}(x) \tag{4.159}$$

under which, H_0 was rejected, for the $p - value < \alpha$, at the significance level of $\alpha = 0.05$. The results of the KS and AD test statistics were presented in Tables 4.26 and 4.27, respectively.

Table 4.26: K-S test results of the fitness of the models to the log returns of the market data

MKT	GBM		NJD		DEJD		ALJD		MDRJD	
	$\hat{K}_{(n,\alpha)}$	p	$\hat{K}_{(n,\alpha)}$	p	$\hat{K}_{(n,\alpha)}$	p	$\hat{K}_{(n,\alpha)}$	p	$\hat{K}_{(n,\alpha)}$	p
NSM	0.115	2.2e-16	0.079	1.0e-10	0.043	6.7e-3	0.031	1.4e-4	0.002	5.8e-1
UKSM	0.064	3.2e-7	0.028	6.3e-5	0.063	5.9e-4	0.003	2.3e-1	0.002	7.3e-1
JSM	0.122	2.2e-16	0.054	0.004	0.038	7.8e-2	0.027	9.8e-2	0.001	5.6e-1

Table 4.26 presents the suitability results of a measure of the suitability of the models to the stock market via the KS method. The results showed that the null hypothesis (H_0), at a significance level $\alpha = 0.05$, was rejected for the GBM and NJD models in all the stock market indices. However, the DEJD model gave a better fit to the empirical stock market data than the GBM and NJD models. The $p-$value obtained in the KS statistic under the JSMI data depicts the non-rejection of H_0. The results also showed that the ALJD and MDRJD models fit the empirical distributions better than the GBM, NJD and DEJD in all the stock market indices. Notwithstanding, H_0 was rejected for the ALJD model in the Nigerian case

Table 4.27: A-D test results of the fitness of the models to the log returns of the market data

MKT	GBM		NJD		DEJD		ALJD		MDRJD	
	$\hat{G}_{(n,\alpha)}$	p	$\hat{G}_{(n,\alpha)}$	p	$\hat{G}_{(n,\alpha)}$	p	$\hat{G}_{(n,\alpha)}$	p	$\hat{G}_{(n,\alpha)}$	p
NSM	0.030	2.5e-4	0.010	1.6e-2	0.008	2.5e-3	0.006	2.5e-2	0.003	5.3e-1
UKSM	0.007	2.5e-4	0.005	1.2e-3	0.004	2.5e-4	0.002	2.4e-1	0.001	3.7e-1
JSM	0.029	2.5e-4	0.007	3.4e-2	0.005	2.5e-4	0.004	1.3e-1	0.001	2.1e-1

Table 4.27 shows the compatibility analysis results of a measure of the fitness of the models to the stock markets. A high rejection rate of the null hypothesis (H_0) was observed in the GBM, NJD and DEJD models having the GBM in the worse-case scenario. The fitness of these models to the empirical was shown by the values of the test statistics: $\hat{G}_{(n,\alpha)}$. According to the values obtained for $\hat{G}_{(n,\alpha)}$, the MDRJD model was found to fit the empirical densities better than others, especially in the JSMI case.

4.9.2 Results of the moments of empirical and modelled distributions

Here, the results of the basic empirical moments (mean, variance, skewness and kurtosis) of the log returns of the data sets and the moments of the distributions of the models were presented in Tables 4.28-4.30

Table 4.28: Results of the moments of empirical log-NASI and modelled distributions

models \ moments	**mean**	**variance**	**skewness**	**kurtosis**
Emp^{NASI}	$2.71e-4$	$1.13e-4$	-0.18	29.31
GBM	$2.71e-4$	$1.13e-4$	0	3
NJD	$2.34e-3$	$1.23e-2$	0.13	6.14
DEJD	$3.36e-2$	$1.23e-2$	-0.17	8.01
ALJD	$8.46e-4$	$3.89e-3$	0.02	9.01
MDRJD	$2.83e-4$	$1.34e-4$	0.13	18.29

The Table 4.28 gives the results of the basic moments of the Nigerian stock market log returns as compared to the moments of the modelled densities. The derived basic moments of ALJD and MDRJD models can be found in subsections (4.6.2) and (4.6.3). The results in the above Table showed that the mean and variance of the GBM model were the same with the empirical NASI, The values obtained under the skewness and kurtosis depict much deviants from the empirical NASI. The distribution of the NJD, ALJD and MDRJD was found to be positively skewed, while the DEJD distribution had a negative skewness. The MDRJD was found to have a better kurtosis compared with the empirical NASI process.

Table 4.29: Results of the moments of empirical log-UKSMI and modelled distributions

models \ moments	**mean**	**variance**	**skewness**	**kurtosis**
Emp^{UKSMI}	$4.82e-4$	$1.06e-2$	-0.56	23.35
GBM	$4.41e-4$	$1.90e-4$	0	3
NJD	$2.32e-2$	$1.86e-2$	-0.19	6.32
DEJD	$1.83e-3$	$3.16e-2$	-0.43	7.13
ALJD	$6.15e-4$	$1.78e-2$	-0.97	13.01
MDRJD	$3.7e-4$	$2.71e-2$	0.17	14.35

The Table 4.29 gives the results of the basic moments of the UK stock market log returns as compared to the moments of the modelled densities. The derived basic moments of ALJD and MDRJD models can be found in subsections (4.6.2) and (4.6.3). The results in the above Table showed that the mean of the GBM model was found to be the same with the empirical UKSMI, the values obtained under the skewness and kurtosis of the GBM depict much deviants from the empirical's skewness and kurtosis. The distributions of the NJD, ALJD and DEJD were found to be negatively skewed, while the MDRJD distribution had a positive skewness. The MDRJD was found to have a better kurtosis compared with the empirical UKSMI process.

Table 4.30: Results of the moments of empirical JSMI and modelled distributions

models \ moments	**mean**	**variance**	**skewness**	**kurtosis**
Emp^{JSMI}	$4.4e-4$	$1.38e-2$	-0.56	11.23
GBM	$1.04e-4$	$6.69e-4$	0	3
NJD	$4.3e-3$	$1.25e-2$	-0.27	5.53
DEJD	$3.11e-3$	$1.42e-2$	-0.57	6.83
ALJD	$6.97e-4$	$2.60e-2$	-0.62	7.52
MDRJD	$3.34e-4$	$1.71e-2$	0.25	9.06

Table 4.30 reports the results of the basic moments of the Japan stock market log returns as compared to the moments of the modelled densities. The derived basic moments of ALJD and MDRJD models can be found in subsections (4.6.2) and (4.6.3). The results in the above Table showed that the mean of the GBM model was found to be the same with the empirical JSMI, the values obtained under the skewness and kurtosis of the GBM depict much deviants from the empirical's skewness and kurtosis. The distributions of the NJD, ALJD and DEJD were found to be negatively skewed, while the MDRJD distribution had a positive skewness. The MDRJD presents a better kurtosis as compared with the empirical JSMI process.

CHAPTER FIVE

DISCUSSION

5.1 Preamble

In this chapter, discussion of all the results obtained in chapter four, in relation to the specific objectives and the existing literature were presented.

5.2 Discussion of results on the asymptotic variances of the particular RMPV processes

Assume that $X_t \in Svsm^j$, such that $X_t = X_t^c + X_t^j$, then the quadratic variation of X_t^j can be obtained by $X_{\Delta,t}^{(1,1)} - X_{\Delta,t}^{(2)}$, which establishes a jump test method according to Barndorff-Nielsen *et al.* (2006c), Barndorff-Nielsen and Shephard (2004), and most recently applied in Gkillas *et al.* (2020b). A generalisation of the above concept has been carried out in this thesis. The jump test models in equations (4.47)-(4.55) for higher-order particular cases of the realised multipower variation process were shown to be alternative and better estimators or measures of jumps than $X_{\Delta,t}^{(1,1)}$ in a discretely-observed data. The asymptotic variances obtained in the particular cases satisfy the inequality:

$$\varphi_{RBV} < \varphi_{RTV} < \varphi_{RQV} < \varphi_{RPV} < \varphi_{RHV} < \varphi_{RH_pV} < \varphi_{ROV} < \varphi_{RDV} < \infty \quad (5.1)$$

This suggest that as m increases in equation (4.46), the value of φ_{RMV} also increases.

5.3 Discussion of results on jump test via the RMPV models in the stock market data

Tables 4.1, 4.2, and 4.3, report the jump test results carried out on the NASI, UKSMI and JSMI data sets which were described in study three of chapter four. From the shreds of evidence produced in Tables 4.1, 4.2 and 4.3, based on a 5% level of significance ($\alpha = 0.05$), the null hypothesis was rejected (in all the particular cases), that the data sets follow a diffusion process $Smsv^c$. Also, based on the results of the p-values and test statistics (Z_m), and the market price process presents jumps. According to the test criterion of rejecting the null hypothesis, H_0, that is, if the $p-val < 0.05$, it was observed from the results of the analysis, that as m increases, the rejection rate of H_0 increases. However, in comparing the jump test results obtained in this work with the empirical analysis of Gkillas et. al (2020b), it was observed that the $p-values$ obtained for the particular higher cases are quite smaller. This implies that the higher-order particular cases of the RMPV processes were better estimators than the bipower variation process used in Barndorff-Nielsen *et al.* (2006c) and Gkillas et. al (2020b). Evidence of jumps in the market price processes were found in Figures 4.7, 4.8, and 4.9.

5.4 Discussion of results on the estimated parameters in the GBM, NJD and DEJD models

The difference in the analysis in this work under the GBM, NJD and DEJD as compared with the estimation of the parameters in the normal and double exponential jump diffusion models in Synowiec (2008), is the initial estimates of the parameters in the models. In the existing literature, arbitrary values were assumed for the values of the parameters in the models when the maximum likelihood estimation is applied. However, in our analysis, the initial estimation of the parameters stems from the stock market process, which gives the actual estimates. This gives the empirical analysis in this thesis an hedge over the methods in literature, owing to the fact that the initials estimates give a true picture of the parameters in the models with regards to the actual stock market price process.

The estimates of the parameters in the GBM, NJD and DEJD models respectively, $\hat{\theta}^{GBM} = (\mu_d, \sigma_d)$, $\hat{\theta}^{NJD} = (\mu_d, \sigma_d, \lambda, \mu_j, \sigma_j)$ and $\hat{\theta}^{DEJD} = (\mu_d, \sigma_d, \eta_1, \eta_2, p, q, \lambda)$ were reported in Tables 4.4- 4.9 under the NASI, UKSMI and the JSMI data. The initial estimates $\hat{\theta}^{int.}$, of the parameters were obtained from the empirical stock data based on the moments of the distributions of the respective models. Also, the optimal parameters $\hat{\theta}^{opm.}$ were obtained via optimising numerically, the log-likelihood of the probability density functions using the RCodes.

The results obtained for the initial and optimal parameters in the GBM model as reported in Tables 4.4 and 4.5 , showed higher mean and volatility of the diffusion process in the UK stock market than the other stock markets under study. However, all the values of the optimal parameters are the same compared to the initial parameters with a slight difference in the optimal means of the UK and Japan markets. This shows that the method used to estimate the optimal parameters is quite robust in the GBM model; this was buttressed by Yang and Aldous (2015). The optimal mean and volatility of the Nigerian stock obtained as contained in Table 4.5 showed different values from the values obtained in Owoloko and Okeke (2014).

In the NJD and DEJD models, the jumps which were found to be present in the stock markets data, via the RMPV jump test method, were incorporated into the GBM model but with different assumptions. In the NJD model, the mean and the volatility of the jump sizes Q_j were estimated as $\hat{\mu}_j$ and $\hat{\sigma}_j$ respectively, such that $Q_j \sim N(\mu_j, \sigma_j)$, and $\hat{\lambda}$ is the estimated jump intensity. The results given in Tables 4.6 and 4.7 with respect to the parameter $\hat{\mu}_j$, show that there are more upward jumps in the Nigerian stock market and more downward jumps in the UK and Japan stock markets. The results of $\hat{\sigma}_j$, indicates that the jumps in the Japan stock are more volatile than the other markets. However, the jump intensity was relatively higher in the Nigerian market than the UK and Japan stock markets.

In the asymmetric DEJD model, the parameters were connected to exponentially distributed processes with means: $\mathbb{E}(Q_i^u) = \frac{1}{\eta_1}$ and $\mathbb{E}(Q_i^d) = \frac{1}{\eta_2}$, given also that the probability of obtaining an upward jump is p and $q = 1 - p$. More so, the intensity of jumps under the asymmetric DEJD is λ as stated in equation (4.131). The Tables 4.8 and 4.9 give the initial and optimal estimates of the parameters in the model. The jump intensity $\hat{\lambda}^{UKSMI}$ is relatively higher than $\hat{\lambda}^{NASI}$ and $\hat{\lambda}^{JSMI}$. This shows that there are more frequent jumps in the UK stock market. The means of the downward jump sizes were found to be greater than the means of the upward jump sizes in the three markets. However, there are more upward jumps in the Nigerian stock market than the UK and Japan stock markets. In the work of Sezgin-Alp (2016), it is observed that the Turkish stock market is less volatile compared to the Nigerian, UK and Japan stock markets under the DEJD model. The values of $\hat{\sigma}^{opm}$ obtained evince this fact.

5.5 Discussion of results on the estimated parameters in the asymmetric Laplace JD model

The independent jump intensities (λ_j^u and λ_j^d) in this work have not been considered in the existing literature (see Synowiec (2008) and Sezgin-Alp (2016)). Here, the downward and upward jump processes were considered independently in the new models. The Tables 4.10 and 4.11 give the results obtained from the estimation of the parameters in the asymmetric Laplace JD model. Here, the occurrence of the upward jumps are modelled separately from the downward jump times. Thus, the upward and downward jump intensities are respectively assumed to be: λ_j^u and λ_j^d, where $\lambda = \lambda_j^u + \lambda_j^d$. The results obtained show that the Nigerian stock market data presents a higher jump intensity than the other two markets. This is further buttressed by the results of p_k and q_k in the three stock markets. It can also be seen that the values of λ are found to be smaller under the ALJD model than the NJD and asymmetric DEJD models above and also in the work of Synowiec (2008). This stems from the dynamics of $Q_j \sim AL(\mu_j, \sigma_j, k)$ in the model.

5.6 Discussion of results on the estimated parameters in the modified double Rayleigh JD model

In Synowiec (2008), standard forms of the jump processes, Q_j^u and Q_j^d were treated. Here, a generalisation of the work of Synowiec by considering a modified version of the double Rayleigh jump-diffusion model was given. One of the motivations for choosing the MDRJD model was that, the densities of Q_j^u and Q_j^d should peak at a non-zero value (according to Synowiec (2008)). That is, $\epsilon_+, \epsilon_- \neq 0$, hence the choice $\epsilon_+ = 0.02$ and $\epsilon_- = -0.02$. The results of the parameters estimated in the MDRJD model were presented in Tables 4.12 and 4.13. The values of the upward and downward jump intensities are found to be small compared with those of the ALJD model. The probabilities p and q, are seen to satisfy the same conditions obtained for p_k and q_k in the ALJD model. The results of the parameters: σ_j^u and σ_j^d looks slightly different (which could be as a result of the generalisation of the model) from the ones obtained in the work of Synowiec (2008). The result in Table 4.13, implies that the densities of Q_j^u under the NASI, UKSMI, and JSMI peaks respectively at 0.0054, 0.0147 and 0.0074. Also, the densities of Q_j^d under the NASI, UKSMI, and JSMI peaks respectively at 0.0519, 0.0370 and 0.057.

5.7 Discussion of results on the comparison of the graphs of the modelled and empirical densities

The work in the existing literature (Synowiec (2008)) compared the graphs of the densities of existing models: GBM, NJD and DEJD etc, to the tails of the distributions of empirical data sets. In this work, new outlooks were presented in this regard. The peakedness and tails of the densities new models were compared with the tails and peakedness of empirical stock market distributions.

The plots of the modelled density functions for the GBM, symmetric NJD, asymmetric DEJD, ALJD, and MDRJD models, with the empirical densities for the NASI, UKSMI and the JSMI were presented in Figures 4.15 - 4.29. It was observed that the peak of the density function of the GBM model, as compared with the empirical densities' peaks was found to be the lowest. The peak of the density function of the symmetric NJD, was better placed than the GBM density. The ALJD and the MDRJD models have the highest peaks that are well fitted to the empirical densities. However, the tails of the NJD and DEJD densities fit better than those of the ALJD and MDRJD densities; showing longer tails to the right than the left in the case of the NJD and the reverse in the DEJD model. The ALJD density presents longer tail to the left in the case of the empirical log returns of the NASI than in the other markets. There was also found a longer tail to the right in the density of the modelled MDRJD for the log returns of the UKSMI.

5.8 Discussion of results on the sensitivity analysis of varied jump-threshold on the parameters in the models

The sensitivity analysis of varied threshold of jumps on the parameters in a family of symmetric and asymmetric jump-diffusion models is new in literature (see Synowiec (2008), Gkillas et. al (2020b), Sezgin-Alp (2016)), to the best of our knowledge. Hence, the sensitivity analysis to obtain the extent to which these parameters respond to the varied jump-threshold were carried out.

The results of the sensitivity analyses on the choice of the threshold of jumps to the parameters in the models were presented in the Tables 4.14-4.25. The results in Tables 4.14, 4.15 and 4.16, show that the estimated optimal parameters remain constant, under the different values of the threshold of jumps. This implies that the method of estimation applied, does not depend on ϵ. Although, the choice of ϵ cannot outrightly be neglected in the sense that, if $\epsilon > X_{\Delta,t}^{max}$, then $\lambda = 0$ and automatically it will not be possible to estimate the parameters with the jump process.

Based on the shreds of evidence produced in Tables 4.17-4.25, it was shown that the optimal parameters are very sensitive to the varied threshold of jumps. Hence, the estimation method strongly depends on the varied threshold of jumps in the DEJD, ALJD and the MDRJD models.

5.9 Discussion of results on the suitability analysis of the models to empirical data

The suitability of the modelled-distributions to the empirical distributions of the stock market indices log returns, was measured via the KS and AD test statistics, as well as by comparing the empirical moments to the basic moments of the modelled-distributions.

The KS results given in the Table 4.26 show that the null hypothesis (H_0), at a significance level $\alpha = 0.05$, was rejected for the GBM and NJD models in all the stock market indices. However, the DEJD model gives a better fit to the empirical stock market data than the GBM and NJD models. The $p-$value obtained in the KS statistic under the JSMI data depicts the non-rejection of H_0. The results also show that the ALJD and MDRJD models fit the empirical distributions better than the GBM, NJD and DEJD in all the stock market indices. Notwithstanding, H_0 was rejected for the ALJD model in the Nigerian case.

The larger KS statistic value obtained for the GBM model, especially in the NASI and JSMI data, depicts a high deviant of the peakedness of the density of the modelled-GBM from the empirical density as could be clearly seen in Figures 4.15 and 4.17. It was also observed that the values of the AD statistics as reported in Table 4.27, are smaller than the KS statistics in Table 4.26. This commemorates the observations made by Synowiec (2008) in a similar analysis. According to Synowiec (2008), the AD statistic gives better picture of the compatibility of the tails of the distributions than the KS statistic.

The null hypothesis, according to the AD statistics was rejected for the GBM, NJD and DEJD models in the NASI, UKSMI and JSMI data. The ALJD and MDRJD model fit the empirical log returns better than the GBM, NJD and DEJD models. However, the ALJD was also rejected in the NASI log returns.

The results of the compatibility analysis via comparing the empirical and the modelled-moments are reported in Tables 4.28, 4.29 and 4.30. They present very high kurtosis and negatively skewed properties for the NASI data; very high negative skewness and kurtosis for the UKSMI and the JSMI data. The results of moments of the NJD, DEJD, ALJD and MDRJD models, in the Nigerian market depict higher kurtosis as compared to that of the GBM model. Although, with the exemption of the DEJD model, in the Nigerian market, the others are positively skewed. The reverse is the case in the skewness property of NJD and ALJD in the UKSMI and JSMI data. In conclusion, the GBM model in all the stock markets gives the worst-case scenario.

CHAPTER SIX

SUMMARY AND CONCLUSIONS

6.1 Preamble

This chapter entails the summary and concluding remarks of the overall research, and enumerates some vital recommendations, contributions to knowledge as well as suggestions for further research.

6.2 Summary

The main determinants of the dynamics of most discretely-observed data are their empirical features and distributional properties. These include non-normality properties such as discontinuous paths (jumps) in most stock indices log returns, asymmetry and high-peakedness in the distribution.

The stock markets are encompassed with numerous challenges such as, the right dynamics to enhance proper prediction of future stock prices, optimality, asset pricing modelling etc. The choice of the dynamics used to represent the trend of market price process can proffer solutions to most of these challenges. This study, therefore, carried out an investigation to detect some non-normality features, so as to detect jumps in a discretely-observed process. Hence, the particular cases of higher-order realised multipower variation process with regards to their asymptotic properties (probability limits and limit distribution) were studied. Models, based on the asymptotic results, for detecting jumps in discretely-observed sampled data from the stock markets were developed.

An improvement in the geometric Brownian motion model when jumps are detected was also carried out. Novel and more robust skewed jump-diffusion models were suggested to cater for the upward and downward jump processes, $J(Q_j^u)$ and $J(Q_j^d)$, respectively. These processes have finite jump activities N_t^u and N_t^d, with their respective jump intensities given as: λ_j^u and λ_j^d, could capture asymmetry and high-peakedness. Also, the processes were assumed to obey the asymmetric

Laplace and the modified double Rayleigh distributions. Hence, the asymmetric Laplace jump-diffusion (ALJD) and the modified double Rayleigh jump-diffusion (MDRJD) models for stock price modelling were proposed.

The probability density functions of the new skewed jump-diffusion processes were obtained via the convolution of densities method and also subject to the condition that the coefficient of the jump process V is a Bernoulli random variable satisfying $\mathbb{P}(V = 1) = \lambda\Delta t$ and $\mathbb{P}(V = 0) = 1-\lambda\Delta t$; where, $\lambda = \lambda_j^u + \lambda_j^d$. The L*évy*-Khintchine formula for obtaining the basic moments was obtained for the ALJD process. Since this research is geared towards obtaining models that are driven by the market trends, were obtained the initial appraisal of the parameters in the models from the log returns of the empirical data sets. The maximum likelihood estimation method was used to obtain the optimal values in the models. The initial and optimal parameters were obtained under varied threshold of jumps, specifically five threshold of jumps. Furthermore, a sensitivity analysis to show the extent to which the optimal parameters respond to the varied threshold of jumps was done. The suitability of five stock models viz: geometric Brownian motion, NJD, DEJD, ALJD and MDRJD models of the stock market empirical data was investigated via the Kolmogorov-Smirnov and Anderson-Darling test statistics, as well as by comparing the basic moments of the empirical data sets with the moments of the modelled distributions.

The methods and models described above were tested on three stock market indices obtained from the web platforms: *https//:www.forextime.com* and the Nigerian stock exchange web platform. The data sets comprise 5334, 2076 and 2076 stock market indices daily observations, respectively, for the Nigerian, UK and Japan stock markets.

The results obtained from the above analysis showed that the higher-order particular cases of the RMPV processes are better estimators of jumps than the bipower variation process used in Barndorff-Nielsen and Shephard (2006). Hence, jumps were vividly seen in the stock market price processes via the RMPV models.

The optimal parameters obtained in the NJD, DEJD, ALJD and the MDRJD models show that the mean of the jump sizes in the price process are greater in the Nigerian stock market price process, than the UK and Japan stock market price process. Also, it was observed that the jumps were more frequent in the UKSMI than the NASI and JSMI. Based on the ALJD and MDRJD models, the intensity of the upward jump process was found to be higher in the Nigerian market than the intensity of the downward jump process; in the UK and Japan stock markets, the reverse was the case.

The comparison between the plots of the modelled and empirical densities, in terms of high-peakedness, showed a worst scenario in the GBM case, and the best in the MDRJD. However, the tails of the density of the NJD model was seen to be well fitted to the empirical densities than the others. The sensitivity analysis results to the varied threshold of jumps showed that the optimal parameters in the NJD models are not sensitive to the varied threshold of jumps, while strong sensitivity was observed in the DEJD, ALJD and MDRJD models.

Finally, the suitability or compatibility analysis results showed that the ALJD and MDRJD models fit the empirical distributions better than the existing models used in literature.

6.3 Conclusion

In conclusion, based on the models, analyses and results obtained in this research, the jump test models of the particular higher-order cases were found to be better jump-estimators. The asymmetric-Laplace and modified double-Rayleigh jump-diffusion models proved more suitable for capturing jumps and non-normal features in the stock market indices.

Although the existing jump-diffusion models conform to some non-normal empirical features found in reality, the novel jump-diffusion models proved better and more robust than the existing ones and can be used for optimality analysis, risk hedging management and future prediction.

6.4 Contributions to knowledge

This research has contributed to existing knowledge in the following ways:

1. The work established a generalised jump test method by introducing the particular higher order cases of the realised multipower variation process in the study of risk management.

2. The assertion of jump components via the realised multipower variation process is new particularly in the Nigerian market. This will be an added advantage to financial analyst in the Nigerian market.

3. The study has also added to extant literature by picturing the asymmetric properties and jump intensities of the jumps separately.

4. The study has presented new dynamics of stock price processes and suggested the asymmetric Laplace and the modified double Rayleigh distributions for the jump processes.

5. The added advantage of the study to market risk managers is that a true picture of jumps engendered by the influx of good and bad news into the stock markets was shown. Notably, the parameters associated with the upward and downward trends evince this.

6.5 Suggestions for further research

Presently, our analysis in this study was restricted to models with constant parameters. This could be extended in future research to modelling the stock market dynamics with models driven by non-constant parameters. This may further strengthen the claim of Areerak (2014) that parameters associated with stock price models are not constants in the actual sense, since they depend on time.

Owing to the closed form solution of the densities and the characteristic functions derived in the jump-diffusion models given in this research, Option pricing can be achieved in the future work, by using some numerical methods or numerical integration techniques.

Furthermore, numerical implementation of Option prices can be done via deriving the partial integro differential equations based on the models in this work.

Appendices

Appendix I: Jump test analysis for the higher-order particular cases of the RMPV processes with Rcodes

```
rm(list = ls()); library(ggplot2); library(dplyr)
jumptest <- function(dp,method=''); #METHOD: Bipower
if(method=='Bipower'){
mu43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
# asymptotic variance
a <- (pi/2)^2 + pi-5; l <- length(dp)   # data length
 rv <- sum(dp^2) # realised variance
# bipower variance (bias adjusted estimator)
bv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
mu43 <- 1/sqrt(pi)*2^(2/3)*gamma((4/3+1)/2)
tp <- mu43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)])*
abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(4/3))
b <- tp/bv^2
# Test statistic
Num <- sqrt(l)*(rv-bv)/rv; Den <-  sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {}; out$asymptoticVar = a;
   out$ztest = z.rjt;
#  out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt)); out$realisedVar = rv;
```

```
out$Bipowervar = bv; return(out); }else if(method=='
   Tripower'){
# tripower variation
mu43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
mu23 <- (1/sqrt(pi))*(2^(1/3))*gamma((2/3+1)/2) #Asym Var
a <- (mu43/(mu23^2))*((mu43^2)/(mu23^4) + 2*(mu43/mu23^2) +
    2) - 7
l <- length(dp)   # data length; rv <- sum(dp^2)
# integrated variance
# bipower variance (bias adjusted estimator)
bv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
rtp <- mu23^(-3)*l/(l-2)*sum(((abs(dp[1:(length(dp)-2)])
*abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(2/3)))
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
tp <- mu43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)])*
abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(4/3))
b <- tp/bv^2
# Test statistic
Num <- sqrt(l)*(rv-rtp)/rv;Den <-  sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {}; out$asymptoticVar = a;
out$ztest = z.rjt; out$critical.value=qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt));
out$realisedVar = rv; out$TripowerVar = rtp;
return(out);} else if(method=='Quadpower'){
# quadpower variation
mu43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
mu1 <- (1/sqrt(pi))*(2^(1/2))*gamma((1+1)/2)
mu12 <- (1/sqrt(pi))*(2^(1/4))*gamma((1/2+1)/2)
a <- (mu1/(mu12^2))*((mu1^3)/(mu12^6) + 2*(mu1^2/mu12^4)
+2*(mu1/mu12^2)+ 2) - 9 # asymptotic variance
l <- length(dp)   # data length
rv <- sum(dp^2)    # realised variance
```

```
# bipower variance (bias adjusted estimator)
bv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
# daily realised tr
rqp <- mu12^(-4)*l/(l-3)*sum(((abs(dp[1:(length(dp)-3)])*
abs(dp[2:(length(dp)-2)])*abs(dp[3:(length(dp)-1)])*
abs(dp[4:length(dp)]))^(1/2)) )
# realised tripower variation with r=s=u=4/3
tp<-mu43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)])*abs(
    dp[2:(length(dp)-1)])*
abs(dp[3:length(dp)]))^(4/3)); b <- tp/bv^2
# Test statistic
Num <- sqrt(l)*(rv-rqp)/rv; Den <-  sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {};
out$asymptoticVar = a; out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt)); out$realisedVar = rv;
out$QuadpowerVar = rqp; return(out)
#hexpower variation
mu43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
mu23 <- (1/sqrt(pi))*(2^(1/3))*gamma((2/3+1)/2)
mu13 <- (1/sqrt(pi))*(2^(1/6))*gamma((1/3+1)/2)
a <- (mu23/(mu13^2))*((mu23^5)/(mu13^10) + 2*(mu23^4/mu13
    ^8)+
2*(mu23^3/mu13^6)+ 2*(mu23^2/mu13^4) +
2*(mu23/mu13^2)+ 2) - 13 # asymptotic variance
l <- length(dp)   # data length
rv <- sum(dp^2)    # realised variance0
# integrated variance
# bipower variance (bias adjusted estimator)
bv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
# daily realised hexapower variation
rhv <- mu13^(-6)*l/(l-5)*sum(((abs(dp[1:(length(dp)-5)])*
abs(dp[2:(length(dp)-4)])*abs(dp[3:(length(dp)-3)])*
```

```
abs(dp[4:(length(dp)-2)])*abs(dp[5:(length(dp)-1)])*
abs(dp[6:length(dp)]))^(1/3))  )
# Integrated quarticity
# realised tripower variation with r=s=u=4/3 (bias adjusted
    estimator)
tp <- mu43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)])*
abs(dp[2:(length(dp)-1)])*
abs(dp[3:length(dp)]))^(4/3))
b <- tp/bv^2
# Test statistic
Num <- sqrt(l)*(rv-rhv)/rv
Den <-  sqrt(a*max(1,b))
z.rjt <- Num/Den
out = {};
out$asymptoticVar = a;
out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.025,0.975));
out$pvalue = 2*pnorm(-abs(z.rjt));
out$realisedVar = rv;
out$HexpowerVar = rhv;
return(out)
 }else if(method=='Heptpvowere r'){
varphi43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
varphi47 <- (1/sqrt(pi))*(2^(2/7))*gamma((4/7+1)/2)
varphi27 <- (1/sqrt(pi))*(2^(1/7))*gamma((2/7+1)/2)
a <- (varphi47/(varphi27^2))*((varphi47^6)/(varphi27^12) +
2*(varphi47^5/varphi27^10)+2*(varphi47^4/varphi27^8)
+ 2*(varphi47^3/varphi27^6) +2*(varphi47^2/varphi27^4)
+ 2*(varphi47/varphi27^2)+2) - 15 # asymptotic variance
l <- length(dp)   # data length
rpv <- sum(dp^2)    # realised variance
# bipower variance (bias adjusted estimator)
bpv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
```

```
# daily realised heptapower variation
rhpv <- varphi27^(-7)*l/(l-6)*sum(((abs(dp[1:(length(dp)-6)
    ])*
abs(dp[2:(length(dp)-5)])*abs(dp[3:(length(dp)-4)])*
abs(dp[4:(length(dp)-3)])*abs(dp[5:(length(dp)-2)])*
abs(dp[6:(length(dp)-1)])*abs(dp[7:length(dp)]))^(2/7)) )
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
tpv <- varphi43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)
    ])*
abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(4/3))
b <- tpv/bpv^2
# Test statistic
Num <- sqrt(l)*(rpv-rhpv)/rpv; Den <-sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {};
out$asymptoticVar = a; out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt)); out$realisedVar = rpv;
out$Heptpvowere rpvar = rhpv; return(out) }else if(method
    =='Octapower'){
varphi43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
varphi12 <- (1/sqrt(pi))*(2^(1/4))*gamma((1/2+1)/2)
varphi14 <- (1/sqrt(pi))*(2^(1/8))*gamma((1/4+1)/2)
#Asymptotic Variance
a <- (varphi12/(varphi14^2))*((varphi12^7)/(varphi14^14) +
2*(varphi12^6/varphi14^12)+2*(varphi12^5/varphi14^10)
+ 2*(varphi12^4/varphi14^8) + 2*(varphi12^3/varphi14^6)
+ 2*(varphi12^2/varphi14^4)+2*(varphi12/varphi14^2) +2) -
    17
l <- length(dp)   # data length
rpv <- sum(dp^2)    # realised variance
# bipower variance (bias adjusted estimator)
bpv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
```

```
# daily realised Octapower variation
rov <- varphi14^(-8)*l/(l-7)*sum(((abs(dp[1:(length(dp)-7)
    ])*
abs(dp[2:(length(dp)-6)])*abs(dp[3:(length(dp)-5)])*
abs(dp[4:(length(dp)-4)])*abs(dp[5:(length(dp)-3)])*
abs(dp[6:(length(dp)-2)])*abs(dp[7:(length(dp)-1)])*abs(dp
    [8:length(dp)]))^(1/4)))
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
tpv <- varphi43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)
    ])*
abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(4/3))
b <- tpv/bpv^2
# Test statistic
Num <- sqrt(l)*(rpv-rov)/rpv; Den <-  sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {};
out$asymptoticVar = a; out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt));
out$realisedVar = rpv; out$Octapowerpvar = rov;
return(out); }else if(method=='Nonpower'){
varphi43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
varphi49 <- (1/sqrt(pi))*(2^(2/9))*gamma((4/9+1)/2)
varphi29 <- (1/sqrt(pi))*(2^(1/9))*gamma((2/9+1)/2)
#Asymptotic Variance
a <- (varphi49/(varphi29^2))*((varphi49^8)/(varphi29^16)+2
*(varphi49^7)/(varphi29^14)+2*(varphi49^6/varphi29^12)+
2*(varphi49^5/varphi29^10)+2*(varphi49^4/varphi29^8) +2*(
    varphi49^3/varphi29^6)+2*(varphi49^2/varphi29^4)+
2*(varphi49/varphi29^2) +2) - 19
l <- length(dp)   # data length
rpv <- sum(dp^2)    # realised variance
# bipower variance (bias adjusted estimator)
```

```
bpv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
# daily realised Nonpower variation
rNv <- varphi29^(-9)*l/(l-8)*sum(((abs(dp[1:(length(dp)-8)
    ])*
abs(dp[2:(length(dp)-7)])*abs(dp[3:(length(dp)-6)])*
abs(dp[4:(length(dp)-5)])*abs(dp[5:(length(dp)-4)])*
abs(dp[6:(length(dp)-3)])*abs(dp[7:(length(dp)-2)])*
abs(dp[8:(length(dp)-1)])*abs(dp[9:length(dp)]))^(2/9)))
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
tpv <- varphi43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)
    ])*
abs(dp[2:(length(dp)-1)])* abs(dp[3:length(dp)]))^(4/3))
b <- tpv/bpv^2; # Test statistic
Num <- sqrt(l)*(rpv-rNv)/rpv; Den <-sqrt(a*max(1,b))
z.rjt <- Num/Den; out = {};
out$asymptoticVar = a; out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt)); out$realisedVar = rpv;
out$Nonpowerpvar = rNv; return(out)
}else if(method=='Decpower'){
varphi43 <- (1/sqrt(pi))*(2^(2/3))*gamma((4/3+1)/2)
varphi25 <- (1/sqrt(pi))*(2^(1/5))*gamma((2/5+1)/2)
varphi15 <- (1/sqrt(pi))*(2^(1/10))*gamma((1/5+1)/2)
#Asymptotic Variance
a <- (varphi25/(varphi15^2))*((varphi25^9)/(varphi15^18)
+2*(varphi25^8)/(varphi15^16)+2*(varphi25^7/varphi15^14)
+2*(varphi25^6/varphi15^12)+2*(varphi25^5/varphi15^10)
+2*(varphi25^4/varphi15^8)+2*(varphi25^3/varphi15^6)
+2*(varphi25^2/varphi15^4)+2*(varphi25/varphi15^2) +2)- 21
l <- length(dp)   # data length
rpv <- sum(dp^2)    # realised variance
# bipower variance (bias adjusted estimator)
```

```
bpv <- pi/2*l/(l-1)*sum(abs(dp[-length(dp)])*abs(dp[-1]))
# daily realised Nonpower variation
rDv <- varphi15^(-10)*l/(l-9)*sum(((abs(dp[1:(length(dp)
-9)])*abs(dp[2:(length(dp)-8)])*abs(dp[3:(length(dp)-7)])*
abs(dp[4:(length(dp)-6)])*abs(dp[5:(length(dp)-5)])*
abs(dp[6:(length(dp)-4)])*abs(dp[7:(length(dp)-3)])*
abs(dp[8:(length(dp)-2)])*abs(dp[9:(length(dp)-1)])*
abs(dp[10:length(dp)]))^(1/5)))
# Integrated quarticity
# realised tripower variation with r=s=u=4/3
tpv <- varphi43^(-3)*l^2/(l-2)*sum((abs(dp[1:(length(dp)-2)
    ])*
abs(dp[2:(length(dp)-1)])*abs(dp[3:length(dp)]))^(4/3))
b <- tpv/bpv^2
# Test statistic
Num <- sqrt(l)*(rpv-rDv)/rpv; Den <-sqrt(a*max(1,b))
z.rjt <- Num/Den
out={}; out$asymptoticVar = a;
out$ztest = z.rjt;
# out$critical.value = qnorm(c(0.05,0.95));
out$pvalue = 2*pnorm(-abs(z.rjt));
out$realisedVar = rpv; out$Decpowerpvar= rDv;
return(out)
#Empirical Analysis
data1 <- read.csv("D:/Back Up/latest project/Analysis")
data1$dates <- format(as.Date(data1$DATES,format = "%d/%m/%
   Y"),"%m/%Y")
days <- c(unique(data1$dates)); Result <- c()
ndays <- length(days); for (i in 1:ndays){
data<- data1[data1[,4]==days[i],]
logreturn<- diff(log(data$CLOSE))
## Jump Test: Choose one of Bipower, Tripower, etc.
jTest <- jumptest(logreturn,method='Decpower')
```

```
Result <- c(jTest , Result)}
Result <- data.frame(matrix(Result , byrow=F,ncol=5,
dimnames=list(days=c(unique(data1$dates)),
Statistics=c("asymptoticVar","ztest","pvalue","realisedVar
    ","Decpowerpvar"))))
Decpower<- data.frame(matrix(unlist(Result), ncol = 5,byrow
     = T,dimnames=list(days=c(unique(data1$dates)),
    Statistics
=c("asymptoticVar","ztest","pvalue","realisedVar","
    Decpowerpvar"))))
Decpower$ID <- seq.int(nrow(Decpower))
Decpower <- Decpower[,c(6,1:5)]
Decpower <- data.frame(Decpower)
ggplot(Decpower)+ geom_line(aes(x=ID,
y=Nonpowerpvar, col='Decpowerpvar'), size =1.5)+
geom_line(aes(x = ID,y =realisedVar,
col = 'realisedVar'),size=0.8)+xlab('Obserpvations') + ylab
    ('') +theme(panel.background =element_blank())+theme(
    legend.title = element_blank())
```

Appendix II: Estimation of the parameters in the models using Rcodes

```
rm(list = ls()); setwd("D:/Back Up/latest project/Analysis
    ")
library(EstimationTools); library(readxl)
library(BBmisc); S <- read_excel("data3.xlsx")
n = length(S$Price);   dt = 1/252
t = seq(0,(n - 1)*dt,length=n); R =diff(log(S$Price))
epislon = 0.02
#if true, we consider the equation value as jump
jumpupindex = which(R > epislon); jumpdownindex = which(R <
     -epislon)
length(jumpupindex)
lambdauhat = length(jumpupindex)/((length(S$Price)-1)*dt)
lambdadhat = length(jumpdownindex)/((length(S$Price)-1)*dt)
```

```
length(jumpdownindex)
lambdahat = lambdauhat +lambdadhat
#diffusion
diffusionindex = which(abs(R) <= epislon)
Rdiffusion = R[diffusionindex]
sigmahat = sd(Rdiffusion)/(sqrt(dt))
muhat = (2*mean(Rdiffusion)+(sigmahat^2)*dt)/(2*dt)
#Probability of upward and downward jumps
pk = length(jumpupindex)/(length(jumpupindex)+length(
    jumpdownindex))
qk = length(jumpdownindex)/(length(jumpupindex)+length(
    jumpdownindex))
k=(1/p-1)^0.5
Rjumpup = R[jumpupindex]
Rjumpdown = R[jumpdownindex]
alpha1hat = 1/mean(Rjumpup)
alpha2hat = 1/abs(mean(Rjumpdown))
mu_jhat= (1/(p*lambdauhat-q*lambdadhat)*dt)*(mean(R) - (
    mu_hat-sigmahat^2/2)*dt)-(lambdauhat*p/eta1hat +
    lambdadhat*q/eta2hat)*dt
#initial Parameter
theta0 <- c(muhat,sigmahat,eta1hat,eta2hat,lambdauhat,
    lambdadhat, p,q,mu_jhat )
theta0
library(knitr); library(rpmutil)
#Integral Function
sigma = sigmahat; mu =muhat
alpha1 = eta1hat; alpha2 = eta2hat
lambdau = lambdauhat; lambdad = lambdadhat
mu_j = mu_jhat; p=p; q=q
Fx = function(r){(((-(r-(mu-sigma^2/2)*dt))-(eta1*sigma^2*
    dt))^2
/((2*sigma^2*dt)*(sqrt(2*pi*sigma^2*dt))))}
```

```
vint <- Vectorize(Fx)
Gx = function(r){(((-(r-(mu-sigma^2/2)*dt)^2)
+(eta2*sigma^2*dt))^2/((2*sigma^2*dt)*(sqrt(2*pi*sigma^2*dt
    )))))}
vint2 = Vectorize(Gx); maxkous <- function(theta, r){
mu <- theta[1]; sigma <- theta(2)
eta1 <- theta[3]; eta2 <- theta[4]
lambdau <- theta[5]; lambdad <- theta[6]
p <- theta[7]; q <- theta[8]; mu_j<- theta[9]
if((lambdau+lambdad) >0){
A = ((1-((lambdau+lambdad)*dt)/sigma*sqrt(dt))*exp(-(r-(mu
    -(sigma^2)/2)*dt)^2)
/(2*sigma^2*dt ))
C =(lambdau*dt*p*eta1*exp(eta1*mu_j)*exp(-(r-(mu-(sigma^2)
    /2)*dt))*eta1)
*exp((sigma^2*eta1*dt)/2)*(integrate(vint, lower=mu_j,
    upper=1000000)$value)
D = (lambdad*dt*q*eta2*exp(eta2*mu_j)
*exp((r-(mu-(sigma^2)/2)*dt))*eta2)
*exp((sigma^2*eta1*dt)/2)*(integrate(vint2,lower=-1000000,
    upper=mu_j)$value)
leftpvart = (C+D); rightpvart = A
maxk= -sum(rightpvart + leftpvart)
}else {
maxk = -sum(log((1/sqrt(2*pi*sigma^2)
*exp(-(r-(mu-sigma^2/2)*dt)^2/(2*(sigma^2*dt))))))
return(maxk)
}
A = optim(theta0,maxkous,r=R); A
mut <- c(A)$par
theta1 <- cbind('mu'=mut[1],'sigma'=mut(2),'eta1'=mut[3],'
    eta2'=mut[4],
'lambdau'=mut[5],'lambdad'=mut[6], "p"=mut[7],"q"=mut[8],"
```

```
    muj"=mut[9])
print(kable(cbind('initial' =c('mu'=muhat,'sigma'=sigmahat
    ,'eta1'=eta1hat,'eta2'=eta2hat,'lambdau'=lambdauhat, '
    lambdad'=lambdadhat, "p"=p,"q"=q,"muj"=muj),'optim'=
    A$par),digits=4))
#Simulated value
logaljdrnd <- function(dt,mu,sigma,eta1,eta2,lambdau,
    lambdad, p,q,muj,t,Ns){
dN = rpois(length(t)-1,abs(lambdau+lambdad)*dt)
dW = sqrt(dt)*rnorm(length(t)-1,0,1)
Y = (((muj+1/eta1)*p - (muj+1/eta2)*q))*dN + (((muj-(2*muj/
    eta1)-(2/eta1^2))*p - (muj^2-(2*muj/eta2)+(2/eta2^2))*q)
    )*dN*runif(length(t)-1,Ns)
Rsim = (mu - sigma^2/2)*dt + (sigma*dW) + Y
}
RSim <- (logaljdrnd(dt,theta0[1],theta0(2),theta0[3],theta0
    [4],theta0[5],theta0[6],theta0[7],theta0[8], theta0[9],t
    ,1))
B = optim(theta0,maxkous,r=RSim)
print(kable(cbind('simu' =c('mu'=mut[1],'sigma'=mut(2),'
    eta1'=mut[3],'eta2'=mut[4],'lambdau'=mut[5], 'lambdad'
    =mut[6], 'p'=mut[7],'q'=mut[8], 'muj'=mut[9]),'optim'=
    B$par),digits=3))
kstest<-ks.test(R,RSim); kstest
library(kdensity); library(EQL)
cdf1 = kdensity(R); cdf2 = kdensity(RSim)
plot(cdf1, main = "",col = "blue",xlim=c(-0.1,0.1), ylim=c
    (0,80),xlab="Log-returns", lwd=2)
lines(cdf2,col = "red", lwd=2)
legend("topleft", legend=c("JSMI","ALJD"),fill=c(1:4))
#install.packages("twosamples")
library(twosamples); adtest<-ad_test(R,RSim)
adtest; adstat<-ad_stat(R,RSim); adstat
```

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